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Heterogeneous Diffusion, Stability Analysis, and Solution Profiles for a MHD Darcy–Forchheimer Model

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Abstract: In the presented analysis, a heterogeneous diffusion is introduced to a magnetohydrodynamics (MHD) Darcy–Forchheimer flow, leading to an extended Darcy–Forchheimer model. The introduction of a generalized diffusion was proposed by Cohen and Murray to study the energy gradients in spatial structures. In addition, Peletier and Troy, on one side, and Rottschäfer and Doelman, on the other side, have introduced a general diffusion (of a fourth-order spatial derivative) to study the oscillatory patterns close the critical points induced by the reaction term. In the presented study, analytical conceptions to a proposed problem with heterogeneous diffusions are introduced. First, the existence and uniqueness of solutions are provided. Afterwards, a stability study is presented aiming to characterize the asymptotic convergent condition for oscillatory patterns. Dedicated solution profiles are explored, making use of a Hamilton–Jacobi type of equation. The existence of oscillatory patterns may induce solutions to be negative, close to the null equilibrium; hence, a precise inner region of positive solutions is obtained.

Keywords: existence; uniqueness; asymptotic; Darcy–Forchheimer; instability

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1. Introduction

The mathematical formulation of non-Newtonian fluids is of relevance to model complex scenarios emerging in engineering and physics. Such a formulation gives rise to additional difficulties when analyzing a general mathematical framework to describe the associated constitutive equations. Unlike in Newtonian fluids, there is not a single constitutive equation that can describe the rheological characteristics of all non-Newtonian fluids. As a consequence, the non-Newtonian fluids are considered under dedicated descriptions subjected to particular applications. This is the case of the Darcy–Forchheimer model that arises in MHD. Other particular applications related with energy, radiation, and convective phenomena of non-Newtonian fluids can be consulted in [1–5].

Non-Newtonian descriptions are typical of fluids in porous medium. Such descriptions apply to different fluid conceptions, depending on their applications: porous catalysis, nuclear reactors cooling, tumor growth dynamics, soil pollution, water movement in reservoirs, oil enhanced recovery, fuel cells, combustion technology, fermentation, or grain storage (see [6–14] for a mathematical approach together with the respective applications).

In the present analysis, we start with the Darcy–Forchheimer second-order flow, given by:

$$\mathbf{V} = (u_1(y,t),0,0) \ \text{ and } \ \nabla\cdot\mathbf{V} = 0 \tag{1}$$

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\frac{\partial^2 u_1}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi v}{K}\right)u_1 - \frac{F}{\rho}u_1^2, \tag{2}$$

$$u_1(y,0) = u_{1,0}(y),$$

where (x, y) represent two classical Cartesian coordinates, P refers to the pressure field, $v = \frac{\mu}{\rho}$ is the kinematic viscosity, F the nonuniform inertia coefficient of porous medium, ρ the density, μ the dynamic viscosity, ϕ the porosity and K the permeability of the medium.

Typically, the diffusive term is associated to a spatial second-order operator derived from the classical linear gradient Fick's law. Nonetheless, some alternative approaches to diffusions were considered to account for heterogeneous solutions with oscillatory characters in the proximity of the equilibrium states. As an example of this approach to biological applications, in [15], the authors study a diffusive structure, accounting for a diffusive gradient. Such a diffusive gradient was obtained from a generalized Landau–Ginzburg free energy model that ends in a fourth-order diffusion. The fourth-order diffusion has been widely considered in different applications, to study heterogeneities in the proximity of the equilibrium solutions. As an example, a similar approach was pursued in [16,17] to introduce a generalization of a classical second-order diffusion to account for oscillatory profiles.

Note that the intention along this analysis is to study some heterogeneous patterns close to the critical points for a MHD flow of the Darcy–Forchheimer type. To this end, a perturbation term, in the form of third-order diffusion, was introduced ad-hoc, so that (2) becomes:

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \epsilon\frac{\partial^4 u_1}{\partial y^3} + v\frac{\partial^2 u_1}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi v}{K}\right)u_1 - \frac{F}{\rho}u_1^2, \tag{3}$$

with $0 < \epsilon << 1$. After making the first derivative with x, the following holds:

$$-\frac{1}{\rho}\frac{\partial^2 P}{\partial x^2} = 0; \quad -\frac{1}{\rho}\frac{\partial P}{\partial x} = K_1.$$

Then (2) becomes:

$$\frac{\partial u_1}{\partial t} = K_1 - \epsilon\frac{\partial^4 u_1}{\partial y^4} + v\frac{\partial^2 u_1}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi v}{K}\right)u_1 - \frac{F}{\rho}u_1^2 \tag{4}$$

$$u_1(y, 0) = u_0(y).$$

Furthermore, other approaches have tried to model heterogeneities in a media via an advection coefficient that precludes the existence of a generalized diffusion (see [18] for a biological interaction application). In [19], the author analyzes the existence of minimal heteroclinic orbits for a class of fourth-order diffusive system with variational terms. In addition, the analysis in [20] develops numerical and analytical approaches to characterize heteroclinic solutions in the traveling wave domain with heterogeneous diffusion for a cooperative system. In the cited analysis, the existence of oscillatory patterns closing the critical points is shown and a characterization is provided. As a consequence, the presented analysis provides insight into the characterization of such oscillatory profiles for an extended Darcy–Forchheimer flow with a non-homogeneous diffusive perturbation.

First, analysis of existence and uniqueness of solutions are provided, based on the use of a normed generalized space shown to be Banach. Afterwards, solutions are shown to be bounded by global initial data over the defined general space. Such solutions are shown to be oscillatory based on a shooting method approach that defines a conservation principle for a Hamiltonian. Solutions are shown to be unique and asymptotic profiles are analytically obtained. Finally, a region of validity for positive solutions with monotone behavior is explored.

2. Analysis of Existence and Uniqueness of Solutions

Firstly, the following generalized norm is defined based on previous similar exercises in [21,22]:

$$\|u_1\|_\rho^2 = \int_R \rho(x) \sum_{k=0}^4 \left|D^k u_1(x)\right|^2 dx, \tag{5}$$

where $D = \frac{d}{dx}$, $u_1 \in H^2_\rho(R) \subset L^1_\rho(R) \subset L^1(R)$ and the weight ρ is defined as :

$$\rho(x) = e^{c_0|x|^{\frac{4}{3}} - \frac{1}{x^q t^a} \int_0^t \left(\left\| \frac{\partial u_1}{\partial x} \right\|^q + 1 \right) ds}. \tag{6}$$

Note that c_0 is a small positive constant and $a > q + 1$.

Lemma 1. *The functional space of functions $u_1 \in H^4_\rho(R) \subset L^1_\rho(R) \subset L^1(R)$ with norm $\|u_1\|_\rho$ is a Banach space.*

Proof. Let $u_1^*, u_1^{**} \in H^4_\rho(R) \subset L^1_\rho(R) \subset L^1(R)$, *then* :

$$\begin{aligned} \|u_1^* + u_1^{**}\|_\rho^2 &= \int_R \rho(x) \sum_{k=0}^4 \left| D^k (u_1^* + u_1^{**})(x) \right| dx \\ &\leq \int_R \rho(x) \sum_{k=0}^4 \left| D^k u_1^*(x) \right| dx + \int_R \rho(x) \sum_{k=0}^4 \left| D^k u_1^{**}(x) \right| dx \\ &= \|u_1^*\|_\rho + \|u_1^{**}\|_\rho. \end{aligned}$$

To prove the space completeness, admit a Cauchy sequence of functions $\{u^*_{n_1}(x) : n_1 \in N\} \in H^4_\rho$ under the norm $\|\cdot\|_\rho$. Suppose that for $\varepsilon \geq 0$, there exists $m \in N$ such that for every $n_1, n_2 > m$, $\left\|u^*_{n_1} + u^*_{n_2}\right\|_\rho \leq \varepsilon$. The convergence is shown as follows:

$$\begin{aligned} \left|u^*_{n_1}(x) - u^*_{n_2}(x)\right| &= \left|\left(u^*_{n_1} - u^*_{n_2}\right)(x)\right| \\ &\leq \left|u^*_{n_1} - u^*_{n_2}\right||x| \\ &\leq \sum_{k=0}^4 \left| D^k \left(u^*_{n_1} - u^*_{n_2}\right)(x) \right| |x| \\ &\leq \rho(x) \sum_{k=0}^4 \left| D^k \left(u^*_{n_1} - u^*_{n_2}\right)(x) \right| |x| \\ &\leq \int_R \rho(x) \sum_{k=0}^4 \left| D^k \left(u^*_{n_1} - u^*_{n_2}\right)(x) \right| |x| dx \\ &= \left\|u^*_{n_1} - u^*_{n_2}\right\|_\rho |x| \leq \varepsilon |x|, \end{aligned}$$

where $\rho(x) \geq 1$. After taking $\varepsilon \to 0$:

$$\left|u^*_{n_1}(x) - u^*_{n_2}(x)\right| \to 0,$$

which shows the convergence of any Cauchy sequence under the norm (5). □

2.1. Primary Assessments

Admit the operator $L = -\epsilon \frac{\partial^4}{\partial x^4} + \nu \frac{\partial^2}{\partial y^2}$ to define the following homogeneous problem:

$$\frac{\partial u_1}{\partial t} = L u_1. \tag{7}$$

Then, the following Lemma is shown:

Lemma 2. *Given $u_0 \in L^1(R)$, then:*

$$\|u_1\|_{L^1} \leq \|u_0\|_{L^1}.$$

Let $u_0 \in H^n(R) \cap L^1(R)$, where $n \in R^+$, then the following inequalities hold:

$$\|u_1\|_{H^n} \leq \|u_0\|_{H^n},$$

$$\|u_1\|_{H^n} \leq \|u_0\|_{L^1}.$$

Furthermore:

$$\|u_1\|_\rho \leq \|u_1\|_{H^n} \leq \|u_0\|_{L^1}.$$

Proof. Admit that the solution of expression (7) is expressed as:

$$u_1(y,t) = e^{tL}u_0(y).$$

After Fourier transformation:

$$\hat{u}_1 = e^{\left(-\epsilon w^4 - \nu w^2\right)t}\hat{u}_0(w).$$

Now, we shall show that $\|u\|_{L^1} \leq \|u_0\|_{L^1}$. For this purpose:

$$\begin{aligned} \|u_1\|_{L^1} &= \int_{-\infty}^{\infty} \left| e^{\left(-\epsilon w^4 - \nu w^2\right)t} \right| |\hat{u}_0(w)| dw \\ &\leq \sup_{w\in R}\{e^{\left(-\epsilon w^4 - \nu w^2\right)t}\} \int_{-\infty}^{\infty} |\hat{u}_0(w)| dw = \|u_0\|_{L^1}. \end{aligned}$$

For $n \in R^+$ and $0 \leq t < \infty$, the following norm is defined (satisfying the A_p-condition for $p = 1$ [23]):

$$\begin{aligned} \|u_1\|_{H^n} &= \int_{-\infty}^{\infty} e^{nw^2} |\hat{u}_1(w)| dw \\ &= \int_{-\infty}^{\infty} e^{nw^2} \left| e^{\left(-\epsilon w^4 - \nu w^2\right)t} \hat{u}_0(w) \right| dw \\ &\leq \sup_{w\in R}\{e^{\left(-\epsilon w^4 - \nu w^2\right)t}\} \int_{-\infty}^{\infty} e^{nw^2} |\hat{u}_0(w)| dw = \|u_0\|_{H^n}. \end{aligned}$$

Let $u_0 \in L^1(R)$, then:

$$\begin{aligned} \|u_1\|_{H^n} &= \int_{-\infty}^{\infty} e^{nw^2} |\hat{u}_1(w)| dw \\ &= \int_{-\infty}^{\infty} e^{nw^2} \left| e^{\left(-\epsilon w^4 - \nu w^2\right)t} \hat{u}_0(w) \right| dw \\ &\leq \sup_{w\in R}\{e^{nw^2 - \epsilon w^4 t - \nu w^2 t}\} \int_{-\infty}^{\infty} |\hat{u}_0(w)| dw. \end{aligned}$$

An elementary assessment leads to:

$$\|u_1\|_{H^n} \leq |\left(\frac{n}{\epsilon t} - \frac{\nu}{\epsilon}\right)^{\frac{1}{2}}| \, \|u_0\|_{L^1},$$

so we have:

$$\|u_1\|_{H^n} \leq \|u_0\|_{L^1},$$

for $t \geq \frac{n}{\nu}$. Finally:

$$\begin{aligned} \|u_1\|_\rho &= \int_R \rho(x) \textstyle\sum_{k=0}^{4} \left| D^k u_1(x) \right| dx \\ &\leq \int_R e^{nx^2} \textstyle\sum_{k=0}^{4} \left| D^k u_1(x) \right| dx \\ &\leq \int_R e^{nx^2} |u_1(x)| dx \leq \|u_1\|_{H^n} \leq \|u_0\|_{L^1}. \end{aligned}$$

□

Assuming now the single parameter (t) representation for expression (7), the following holds:

$$Q(y,t) = e^{-\epsilon\Delta^2 t}. \tag{8}$$

For $t > 0$, the operator $\epsilon\Delta^2$ is the infinitesimal generator of a strongly continuous semigroup, so that the following abstract evolution holds:

$$\begin{aligned} u_1(t) = e^{-\epsilon\Delta^2 t}u_0 \quad &+ \int_0^t \Big[K_1 e^{-\epsilon\Delta^2(t-s)} + \nu\Delta e^{-\epsilon\Delta^2(t-s)} \\ &- e^{-\epsilon\Delta^2(t-s)}u(s)\left(\tfrac{\sigma B_0^2}{\rho} + \tfrac{\phi\nu}{K} + \tfrac{F}{\rho}u(s)\right)\Big]ds. \end{aligned} \tag{9}$$

After application of the Fourier transformation on expression (7) with $u(y,0) = \delta(y)$, the following holds:

$$\hat{u} = e^{-\epsilon w^4 t}\hat{u}_0(w).$$

To obtain a fundamental solution, the following kernel for expression (7) is obtained as:

$$Q(y,t) = F^{-1}\left(e^{-\epsilon w^4 t}\right) = \frac{1}{2\pi}\int_R e^{-\epsilon w^4 t - iwy}dw = \int_R e^{-\nu w^2 t}\cos(wy)dw, \tag{10}$$

by using integration with respect to w over R, it is possible to conclude on the existence of a finite mass kernel. Consequently, we can rewrite the abstract evolution in (9) in terms of such kernel. To this end, consider the following operator in $H^4_\rho(R)$:

$$T_{u_0,t} : H^4_\rho(R) \to H^4_\rho(R),$$

defined as

$$\begin{aligned} T_{u_0,t}(u) \quad &= \quad Q(y,t) * u_0(y) + \int_0^t [K_1 Q(y,t-s) + \nu\Delta Q(y,t-s) * u(y,s)]ds \\ &\quad - \int_0^t Q(y,t-s) * u(s)\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u(s)\right)ds, \end{aligned} \tag{11}$$

so that the following lemma holds.

Lemma 3. *The operator $T_{u_0,t}$ of single parameter (t), is bounded in $H^4_\rho(R)$ with the norm (5)*

Proof. Firstly, we need to show the following inequality:

$$d_0\|u_0\|_\rho \le \|u_1\|_\rho.$$

To this end:

$$\begin{aligned} \|u_1\|_\rho \quad &= \quad \int_R \rho(w)\textstyle\sum_{k=0}^4 \left|D^k\hat{u}_1(w)\right|dw \\ &= \quad \int_R \rho(w)\textstyle\sum_{k=0}^4 \left|D^k\left(e^{(-\epsilon w^4 - \nu w^2)t}\hat{u}_0\right)\right|dw \\ &\ge \quad \int_R \rho(w)\textstyle\sum_{k=0}^4 \left|D^k e^{(-\epsilon w^4 - \nu w^2)t}\right| \textstyle\sum_{k=0}^4 \left|D^k\hat{u}_0\right|dw \\ &\ge \quad d_0\int_R \rho(w)\textstyle\sum_{k=0}^4 \left|D^k\hat{u}_0\right|dw = d_0\|u_0\|_\rho, \end{aligned}$$

where

$$a_0 = \inf_{x\in B_r}\left\{\rho(w)\textstyle\sum_{k=0}^2 \left|D^k e^{(-\epsilon w^4 - \nu w^2)t}\right|\right\} > 0,$$

and $r > 0$ is sufficiently small in $B_r = \{w, |w| < r\}$.

Now, considering the operator $T_{u_0,t}$, the following holds:

$$\begin{aligned}\|T_{u_0,t}(u_1)\|_\rho &\le \|T_{u_0,t}\|_\rho\|u_1\|_\rho \\ &\le \|Q\|_\rho\|u_0\|_\rho + \int_0^t \Big[K_1\|Q\|_\rho + \nu\|\Delta Q\|_\rho\|u_1\|_\rho\Big]ds \\ &\quad + \int_0^t \|Q\|_\rho\|u_1\|_\rho\left\|-\frac{\sigma B_0^2}{\rho} - \frac{\phi\nu}{K} - \frac{F}{\rho}u_1\right\|_\rho ds \\ &\le \left[\|Q\|_\rho\frac{1}{d_0 t} + \int_0^t\left[\frac{K_1\|Q\|_\rho}{d_0 t\|u_0\|_\rho} + \nu\|\Delta Q\|_\rho\right]ds\right]t\|u\|_\rho \\ &\quad + \left[\int_0^t \|Q\|_\rho\left(\left\|\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right\|_\rho - \frac{F d_0}{\rho}\|u_0\|_\rho\right)\right]t\|u\|_\rho,\end{aligned}$$

which implies that

$$\|T_{u_0,t}(u)\|_\rho \le \left[\|Q\|_\rho\frac{1}{d_0 t} + \int_0^t\left[\frac{K_1\|Q\|_\rho}{d_0 t\|u_0\|_\rho} + \|Q\|_\rho\left(\left\|\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right\|_\rho - \frac{F b_0}{\rho}\|u_0\|_\rho\right)\right]ds\right]t. \tag{12}$$

This last inequality permits showing the operator boundness for each value of $t > 0$. □

2.2. Existence and Uniqueness of Solutions

To analyze the existence of solutions to expression (4), a step like initial condition is admitted:

$$u_0(y) = H(-y), \tag{13}$$

where H is the Heaviside step function. We can justify the choice of a Heaviside step function to study the asymptotic behavior of solutions when $y \to \infty$ as $H(-y) = 0$.

The following lemma, based on a shooting method approach, is shown to account for the existence of solutions analysis.

Lemma 4. *Oscillatory solutions to expression* (4) *with the Heaviside initial condition* (13) *do exist.*

Proof. Admit the following Navier pseudo-boundary conditions at $|y| \to \infty$:

$$u_1(|y| \to \infty, t) = u_1''(|y| \to \infty, t) = 0, \tag{14}$$

so that the first and third derivatives are given by two parametric conditions:

$$\begin{aligned} u_1'(|y| \to \infty, t) &= \alpha \in R, \\ u_1'''(|y| \to \infty, t) &= \beta \in R. \end{aligned} \tag{15}$$

Consider the stationary equation of (4):

$$-\epsilon\frac{\partial^4 u_1}{\partial y^4} + v\frac{\partial^2 u}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)u - \frac{F}{\rho}u^2 = 0, \tag{16}$$

where $K_1 = 0$, as the effect of pressure gradient is neglected at $y \to \infty$. A Hamiltonian for the stationary equation is defined as (see [24]):

$$H(u) = \epsilon u_1''' u_1' - \left(\frac{1}{2} - \nu\right)(u_1')^2 - \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)\frac{u_1^2}{2} - \frac{F}{3\rho}u_1^3 + K_2. \tag{17}$$

Since Hamiltonian is an energy functional, it must satisfy the null condition when $|y| \to \infty$, so that:

$$\underset{|y|\to\infty}{Lim} H(u_1(y), u_1'(y), u_1''(y), u_1'''(y)) = 0.$$

Note that $u_1 = 0$ and $u_1 = \frac{\rho}{F}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)$ are constant solutions of expression (16). For the sake of simplicity, we make use of $u_1 = \frac{\rho}{F}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)$ in expression (16) to obtain:

$$K_2 = \frac{\rho^2}{6F^2}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3.$$

Therefore expression (17) becomes:

$$H(u) = \epsilon u_1''' u_1' - \left(\frac{1}{2} - \nu\right)(u_1')^2 - \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)\frac{u_1^2}{2} - \frac{F}{3\rho}u_1^3 + \frac{\rho^2}{6F^2}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3. \quad (18)$$

Any stationary solution, either $u_1 = 0$ or $u_1 = \frac{\rho}{F}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)$ preserve the Hamiltonian. In addition, these stationary constant solutions represent the energy state of any orbit acting asymptotically to such equilibrium solutions. Consequently and after applying the Navier pseudo boundary conditions on Hamiltonian expression (18), the following holds:

$$\alpha = -\frac{\rho^2\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3}{6F^2\epsilon\beta}. \quad (19)$$

Given the Navier conditions expressed, it is possible to conclude on the opposite sign for the first and third derivatives. Additionally, any solution shall satisfy:

$$\underset{y\to\infty}{Lim}(u_1(y), u_1'(y), u_1''(y), u_1'''(y)) = (0,0,0,0).$$

To show the existence of oscillations close the stationary, we define the localization variable ζ given by:

$$\zeta(\alpha) = \sup\{y > 0, u_1' < 0 \text{ in } (0, y)\}.$$

In addition we define:

$$\alpha^* = \sup\{u_1', u_1(\alpha, \beta(\alpha), \zeta(\alpha)) < 1\}, \quad \beta^* = \sup\{u_1''', u_1(\alpha, \beta(\alpha), \zeta(\alpha)) < 1\}.$$

The intention is to obtain a finite value of $\zeta(\alpha)$. To this end, admit α of the form:

$$\alpha = -\frac{1}{\zeta(\alpha)}. \quad (20)$$

The negative sign in expression (20) shows that for the supreme value of α, the maximum value of ζ is achieved. Now, using expression (20), into (19), the following holds:

$$-\frac{1}{\zeta(\alpha)} = -\frac{\rho^2\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3}{6F^2\epsilon\beta}.$$

Taking $\alpha = \alpha^*$ and $\beta = \beta^*$, then the above expression can be written as:

$$\zeta(\alpha^*) = \frac{6F^2\epsilon\beta^*}{\rho^2\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3}. \quad (21)$$

Any heteroclinic orbit between $u_1 = 0$ and $u_1 = 1$ shows that β^* is finite and the stationary solutions satisfy:

$$\begin{aligned} \underset{y\to\infty}{Lim}\left(u_1(y), u_1'(y), u_1''(y), u_1'''(y)\right) &= (0,0,0,0), \\ \underset{y\to\infty}{Lim}\left(u_1(y), u_1'(y), u_1''(y), u_1'''(y)\right) &= \left(\frac{\rho}{F}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right), 0, 0, 0\right). \end{aligned}$$

Then, any non-trivial continuous solution shall have a maximum. Therefore β^* (which represent third derivative) shall be finite. Consequently $\zeta(\alpha^*)$ is finite as well. The same process shall be followed to obtain another locating variable satisfying $y > \zeta(\alpha^*)$, so that the first derivative is positive in the interval $(\zeta(\alpha^*), y)$. Based on this, we define:

$$\eta(\alpha) = \sup\{(y - \eta(\alpha^*)) > 0\,,\, u_1'(\alpha, \beta(\alpha), \cdot) > 0 \text{ in } (\zeta(\alpha^*), y)\},$$

and

$$\alpha^{**} = \inf\{u_1'\,,\, u_1(\alpha, \beta(\alpha), \eta(\alpha)) > 0\}, \quad \beta^{**} = \inf\{u_1'''\,,\, u_1(\alpha, \beta(\alpha), \eta(\alpha)) > 0\}.$$

We can choose the value of α in such a way that the orbit is non-decreasing in the interval $(\zeta(\alpha^*), y)$. This implies that the value of α is positive, and by the expression (20), the value of β is negative. Assuming the finite step function δ in interval $(\zeta(\alpha^*), y)$, the following holds:

$$\alpha = \frac{\delta}{\eta - \zeta}.$$

Making use of the expression (20):

$$\frac{\delta}{\eta - \zeta}\beta = -\frac{\rho^2\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3}{6F^2\epsilon\beta},$$

which implies that

$$\eta = \zeta - \frac{6F^2\epsilon\delta\beta^2}{\rho^2\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3}.$$

To get the highest value of η, consider $\alpha = \alpha^{**}$, then we obtain the infimum of the possible rates of growth. Further, admit the infimum value of the third derivative β^{**} so that the following finite maximum value of η in spatial location holds:

$$\eta(\alpha^{**}) = \zeta - \frac{6F^2\epsilon\delta(\beta^{**})^2}{\rho^2\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)^3}.$$

This last expression shows that the stationary orbits, close to the solutions $u_1 = 0$ and $u_1 = -\frac{\rho}{F}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K}\right)$, have non-increasing behavior and non-decreasing conditions in spatial intervals $(0, \zeta(\alpha^*))$ and $(\zeta(\alpha^*), \eta(\alpha^{**}))$, respectively, where $\zeta(\alpha^*)$ and $\eta(\alpha^{**})$ are finite. Repeating the same process permits concluding on the presence of oscillating orbits closing the critical points. □

2.3. Uniqueness

For the uniqueness analysis, it is required to show that $T_{u_0,t}$ (defined in (11)) has a unique fix point $u_1(y,t) = T_{u_0,t}(u_1(y,t))$. For this purpose:

$$\begin{aligned}
&\|T_{u_{0,t}}(u_1^*) - T_{u_{0,t}}(u_1^{**})\|_\rho \leq \int_0^t \|\nu\Delta Q(y,t-s) * (u_1^* - u_1^{**}) \\
&-Q(y,t-s) * \left[u_1^*\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^*\right) - u_1^{**}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^{**}\right)\right]\Bigg\|_\rho ds \\
&\leq \int_0^t \Bigg\| \int_s^t \nu\Delta Q(y,t-s)(u_1^* - u_1^{**}) \\
&-Q(y,t-s-r)\left[u_1^*\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^*\right) - u_1^{**}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^{**}\right)\right]dr\Bigg\|_\rho ds \\
&\leq \int_0^t \int_s^t \|\nu\Delta Q(y,t-s)(u_1^* - u_1^{**}) \\
&-Q(y,t-s-r)\left[u_1^*\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^*\right) - u_1^{**}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^{**}\right)\right]dr\Bigg\|_\rho ds \\
&\leq \int_0^t \int_s^t \|\nu\Delta Q(y,t-s)(u_1^* - u_1^{**})\|_\rho drds \\
&+\int_0^t \int_s^t \|Q(y,t-s-r)\|_\rho \Bigg\|u_1^*\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^*\right) \\
&-u_1^{**}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1^{**}\right)\|_\rho drds \leq \int_0^t \int_s^t \|\nu\Delta Q(y,t-s-r)\|_\rho \|u_1^* - u_1^{**}\|_\rho drds \\
&+\int_0^t \int_s^t \|Q(y,t-s-r)\|_\rho \left\|\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} - \frac{F}{\rho}(u_1^* + u_1^{**})\right\|_\rho \|u_1^* - u_1^{**}\|_\rho drds.
\end{aligned} \tag{22}$$

Note that from expression (10), Q and ΔQ are bounded. Therefore we can choose:

$$A = \sup\left\{ \|\nu\Delta Q\|_\rho,\ \|Q\|_\rho \left\| \frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} - \frac{F}{\rho}(u_1^* + u_1^{**}) \right\|_\rho ;\ \forall t > 0, y \in R \right\}$$

Then, the following holds:

$$\begin{aligned}
\left\|T_{u_{0,t}}(u_1^*) - T_{u_{0,t}}(u_1^{**})\right\|_\rho &\leq 2A\int_0^t \int_s^t \|u_1^* - u_1^{**}\|_\rho drds \\
&= 2At(t-s)\|u_1^* - u_1^{**}\|_\rho.
\end{aligned}$$

For any ball centered in t, and with radius proportional to $t-s$, the uniqueness is proved in the limit with $u_1 \swarrow u_2$, which provides a contractive mapping $T_{u_0,t}$ in the defined space H_ρ^4.

3. Solution Profiles

Solution profiles are obtained based on the conservation of the Hamiltonian as introduced in Lemma 4. When the Hamiltonian is preserved, the proposed equation can be expressed in the complex plane, keeping the same conservation principle. This approach has been widely followed to obtain a solution for a Schrödinger equation in physics. The fact of operating in the complex space is of relevance to solve the proposed Equation (4) as any behavior exhibited (either oscillatory or monotone) can be treated, making use of the complex exponential notation (see [25] for a complete discussion). Similar approaches have been followed in [26–28]. In addition, in these last references, the authors develop the whole formalism of the exponential scaling. The extension of the exponential complex function into generalized diffusive problems has been tracked in [29] and further discussed to blow up profiles in [30]. Based on the cited references, the following scaling is considered:

$$u = e^{w_1}, \tag{23}$$

so that w_1 satisfies the following Hamilton–Jacobi equation:

$$H_4(w_1) = -\epsilon\left(\frac{\partial w_1}{\partial y}\right)^2\left(\frac{\partial w_1}{\partial y}\right)^2 + \nu\frac{\partial w_1}{\partial y}\frac{\partial w_1}{\partial y} + \left(K_1 - \frac{\sigma B_0^2}{\rho} - \frac{\phi\nu}{K}\right) - \frac{F}{\rho}e^{w_1}. \tag{24}$$

Considering only the leading terms and after replacement in (4):

$$\frac{\partial w_1}{\partial t} = -\epsilon\left(\frac{\partial w_1}{\partial y}\right)^2\left(\frac{\partial w_1}{\partial y}\right)^2 + \nu\frac{\partial w_1}{\partial y}\frac{\partial w_1}{\partial y} + \left(K_1 - \frac{\sigma B_0^2}{\rho} - \frac{\phi\nu}{K}\right) - \frac{F}{\rho}e^{w_1}.$$

Note that existence and uniqueness of solutions to a Hamilton–Jacobi equation extended to non-homogeneous diffusion were analyzed in [29,30]. Based on this, admit that a solution to (4) can be expressed as separated variables:

$$w_1(y,t) = (\tau + t)^{-\frac{1}{3}}\eta(y), \tag{25}$$

where $\tau < t < T$. In the asymptotic approach with $t \to \infty$, it holds that $\frac{F}{\rho}e^{w_1} = \frac{F}{\rho}e^{(\tau+t)^{-\frac{1}{3}}\eta(y)} \to \frac{F}{\rho}$. Introducing (25) into (24):

$$-\frac{1}{3}\eta = -\epsilon\eta_y^4 + \nu(\tau+t)^{\frac{2}{3}}\eta_y^2 + \left(K_1 - \frac{\sigma B_0^2}{\rho} - \frac{\phi\nu}{K} - \frac{F}{\rho}\right),$$

where $t \to \infty$. Again, considering the leading terms:

$$-\frac{1}{3}\eta = \nu(\tau+t)^{\frac{2}{3}}\eta_y^2 + \left(K_1 - \frac{\sigma B_0^2}{\rho} - \frac{\phi\nu}{K} - \frac{F}{\rho}\right).$$

Solving by standard separation of variable techniques, we obtain:

$$\eta(y) = 3\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} - \frac{F}{\rho} - K_1\right) - \frac{1}{12\nu^2(\tau+t)^2}y^2.$$

In the asymptotic approximation, t is large enough so as to take $\eta \to 0$. Consequently, the above expression can be written as:

$$y = 6\nu t\sqrt{\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} - \frac{F}{\rho} - K_1}.$$

Balancing the first derivatives $\eta_y^2 << (\tau + t)$, the following holds:

$$-\frac{1}{3}\eta = -\epsilon\eta_y^4 + \left(K_1 - \frac{\sigma B_0^2}{\rho} - \frac{\phi\nu}{K} - \frac{F}{\rho}\right).$$

After solving:

$$\eta(y) = 3\left(\frac{1}{4\epsilon}y\right)^{\frac{4}{3}} + 3\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho} - K_1\right).$$

Setting the value of $\eta(y)$ into (25):

$$w_1(y,t) = 3t^{-\frac{1}{3}}\left[\left(\frac{1}{4\epsilon}y\right)^{\frac{4}{3}} + \left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho} - K_1\right)\right].$$

After using (23):

$$u_1(y,t) = e^{3t^{-\frac{1}{3}}\left[\left(\frac{1}{4\epsilon}y\right)^{\frac{4}{3}}+\left(\frac{\sigma B_0^2}{\rho}+\frac{\phi\nu}{K}+\frac{F}{\rho}-K_1\right)\right]}.$$

The obtained solution profile to u_1 behaves monotonically. The intention now is to determine a region in the domain to ensure that such monotone behavior holds.

Assessment of a Region with Positive Solutions

The objective in this section is to determine a ball region expressed as $|y| \in B_{k(t)}$, such that in the inner domain no instabilities occur, i.e., solutions are purely monotone. To this end, the following lemma holds:

Lemma 5. *There exists a ball-region $B_{k(t)}$ such that for $|y| << k(t)$ with $k(t) > t^{\frac{1}{4}}|\ln t|$, any solution is purely monotone, i.e., solutions do not exhibit oscillatory behavior.*

Proof. Let us consider the following variable scaling

$$x = \frac{y}{t^{\frac{1}{4}}}; \quad \tau = \ln t \to -\infty \quad \text{if } t \to 0^+. \tag{26}$$

As previously expressed, the effect of the pressure gradient is neglected when $x \to \infty$. As a consequence, the expression (4) reads:

$$\frac{\partial u_1}{\partial \tau} = \left(\mathbf{B} - \frac{1}{4}\mathbf{I}\right)u_1 + \nu e^{\frac{1}{2}\tau}\frac{\partial^2 u_1}{\partial x^2} - e^{\tau}u_1\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_1\right), \tag{27}$$

where $\mathbf{B} = -\epsilon\frac{\partial^4}{\partial x^4} + \frac{1}{4}x\frac{\partial}{\partial x} + \frac{1}{4}\mathbf{I}$.

Consider that any stationary solution is expressed as:

$$\left(\mathbf{B} - \frac{1}{4}\mathbf{I}\right)u_{1e} = 0, \quad u_{1e}(\infty) = 0, \quad u_{1e}(-\infty) = 1. \tag{28}$$

Then, any solution to (27) close the stationary is given by:

$$u_1(x,\tau) = u_{1e}(x) + L(x,\tau). \tag{29}$$

Note that in the proximity of the stationary solution $|L| << 1$. Making use of (29) into (27), the following holds:

$$L_\tau = \left(\mathbf{B} - \frac{1}{4}\mathbf{I}\right)L + \nu e^{\frac{1}{2}\tau}\frac{\partial^2 L}{\partial x^2} + \nu e^{\frac{1}{2}\tau}\frac{\partial^2 u_{1e}}{\partial x^2}\nu e^{\frac{1}{2}\tau}\frac{\partial^2 u_{1e}}{\partial x^2} - e^{\tau}u_{1e}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_{1e}\right). \tag{30}$$

Admit the following asymptotic separation of variables:

$$L(x,\tau) = f(x)g(\tau). \tag{31}$$

Putting (31) in (30), the following holds:

$$\frac{g'(\tau)}{g(\tau)} = \frac{\left(\mathbf{B} - \frac{1}{4}\mathbf{I}\right)f + \nu e^{\frac{1}{2}\tau}\frac{d^2 f}{dx^2} + \frac{\nu e^{\frac{1}{2}\tau}}{g(\tau)}\frac{\partial^2 u_{1e}}{\partial x^2} + \frac{e^{\tau}u_{1e}}{g(\tau)}\left(\frac{\sigma B_0^2}{\rho} + \frac{\phi\nu}{K} + \frac{F}{\rho}u_{1e}\right)}{f} = \beta, \tag{32}$$

which implies that

$$g(\tau) = e^{\tau}, \tag{33}$$

for simplicity, we take $\beta = 1$.

To find a solution for f, we apply the asymptotic condition $u_{1e}(\infty) = 0$ in (32):

$$\left(\mathbf{B} - \frac{1}{4}\mathbf{I}\right) f + +\nu e^{\frac{1}{2}\tau} \frac{d^2 f}{dx^2} = f. \tag{34}$$

As exposed in the previous section and based on a solution proposal in [29], a solution to (34) can be expressed as:

$$f(x) = e^{\alpha x}. \tag{35}$$

Introducing the last expression into (34) and after balancing the leading terms:

$$\alpha^4 = -\frac{1}{\epsilon}. \tag{36}$$

Provided that

$$\frac{1}{4}x + \nu e^{\frac{1}{2}\tau} << 1. \tag{37}$$

Note that, $\nu > 0$ as it represents the kinematic viscosity. Therefore:

$$\frac{1}{4}x << 1,$$

which implies that

$$t \geq \frac{1}{4}|y|. \tag{38}$$

This last expression shows the validity region for the exponential representation in (35). Finally, the solutions given by the two main real roots of α are given by:

$$f_+ = e^{\alpha x}, \ \ x \to -\infty; \ \ f_- = e^{-\alpha x}, \ \ x \to \infty.$$

So that:

$$L(x, \tau) = e^{\tau}\left(e^{\alpha x} + e^{-\alpha x}\right). \tag{39}$$

The expression (39) becomes:

$$u_1(x, \tau) = u_{1e}(x) + e^{\tau}\left(e^{\alpha x} + e^{-\alpha x}\right).$$

Upon recovery of the original variables (y, t):

$$u_1(y, t) = u_{1e}\left(\frac{y}{t^{\frac{1}{4}}}\right) + t\left(e^{\alpha \frac{y}{t^{1/4}}} + e^{-\alpha \frac{y}{t^{1/4}}}\right).$$

In the asymptotic approach, $y \to \infty$ then $|L| << 1$, therefore:

$$\left| t e^{-\alpha \frac{y}{t^{1/4}}} \right| << 1,$$

which implies that:

$$|y| >> t^{\frac{1}{4}} \ln t.$$

As $\ln t < 0$, then:

$$|y| << t^{\frac{1}{4}} \ln t = k(t). \tag{40}$$

Combining expressions (38) and (40), the following holds:

$$|y| < 4t << t^{\frac{1}{4}} \ln t,$$

for $t \to 0^+$. □

Finally, the same assessment can be followed for any $t = t_0 > 0$ with the re-scaling $\tau = \ln(t - t_0)$. Hence, given any $t = t_0 > 0$, the inner region, where positivity in the solution holds, is defined as:

$$|y| << (t - t_0)^{\frac{1}{4}} |\ln(t - t_0)|.$$

A simple estimation can be obtained assuming that $t \sim 2t_0$ for t_0 is sufficiently small:

$$|y| << t_0^{\frac{1}{4}} |\ln t_0|.$$

4. Conclusions

The proposed generalized diffusion to a classical second-order Darcy–Forchheimer flow was treated in the presented study, leading to an fourth-order operator, defining the extended Darcy–Forchheimer flow. The main advantage of the introduction of a generalized diffusion is to account for oscillatory patterns close the equilibrium conditions induced by the reaction terms. Such patterns have been shown to exist, and even when being important features of the postulated fourth-order operators, they provided uniqueness and stability of solutions given certain condition in the initial data. Afterwards, solution profiles were obtained with asymptotic expansion leading to a Hamilton–Jacobi equation. Such equation was solved based on separation of variable techniques leading to a monotone solution. Finally, as monotone solutions do not hold close the critical points (due to the oscillations induced), a region of positive and, hence, validity for such monotone solutions, was precisely determined.

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