Journal of Applied Analysis and Computation
Volume 12, Number 1, February 2022, 179–205

Website:http://www.jaac-online.com
DOI:10.11948/20210096

NON-HOMOGENEOUS REACTION IN A NON-LINEAR DIFFUSION OPERATOR WITH ADVECTION TO MODEL A MASS TRANSFER PROCESS

Jose Luis Diaz Palencia[1,†], Federico Prieto Munoz[1] and Juan Miguel Garcia-Haro[1]

Abstract It is the objective to provide a mathematical treatment of a non-homogeneous and non-lipschitz reaction problem with a non-linear diffusion and advection operator, so that it can be applied to a fire extinguishing process in aerospace. The main findings are related with the existence and characterization of a finite propagation support that emerges in virtue of the the non-linear diffusion formulation. It is provided a precise assessment on different times associated to the extinguisher discharge process. Particularly, the time required to activate the discharge, the time required for the extinguisher front to cover the whole domain, the time required to reach a minimum level of concentration so as to extinguish a fire and the time required by the agent to reach some difficult dead zones where the extinguisher propagates only by diffusion and no advection. The equation proposed is firstly discussed from a mathematical perspective to find analytical solutions and propagating profiles. Afterwards, the application exercise is introduced.

Keywords Reaction, non-linear advection, non-linear diffusion, non-homogenous, aerospace.

MSC(2010) 35K55, 35K57, 35K59, 35K65.

1. Introduction

An aircraft is equipped with a fire suppressant system whose intention is to extinguish any generated fire. The regulations, involved in the design process, require the fire to be suppressed within a maximum elapsed time so that the aircraft safety is kept during the operation. The typical models, used to predict the dynamic of the fire suppressant agent, are based on numerical algorithms in CFD and a set of testing campaigns to validate the numerical results. In some cases, the use of simple modelling equations leads to significant testing activities (as examples see [4] and [26]). Some designers consider that an extensive testing campaign, in a real platform, is enough to characterize the dynamic of the fire suppressant in each operational condition. Nonetheless, this is a costly task and, further, shall be supported by the construction of an empirical reliable model to predict the suppressant

†The corresponding author. E-mail: joseluis.diaz@ufv.es(J. L. Diaz)
[1]Escuela Politecnica Superior, Universidad Francisco de Vitoria, Ctra. Pozuelo-Majadahonda Km 1,800, 28223, Pozuelo de Alarcon, Madrid, Spain

concentration in any other condition.

Fire suppressant process typical models are based on a linear order two diffusion [13]. In contrast, the use of a non-linear diffusion may produce further accurate results specially related with the existence of a propagating suppressant front with finite speed of propagation. The use of non-linear diffusion is an oblique topic and has been used in other applications where numerical and purely analytical approaches have been followed. Such non-linear diffusion allows to model accurately physical phenomena in which porosity is a governing parameter. The non linear diffusion and the mathematical property related with propagation, as proposed along this paper, has not been previously treated to model a fire extinguisher process in aircrafts. Nonetheless, it has been applied to other studies. The following discussion introduces some references of interest in virtue of the obtained results impact. In [15], the authors develop a model to the spread of wildfires that consider different mechanisms at different scales that drive the fire process in wildlands. The proposed set of equations are based on an energy balance to derive the classical diffusion temperature equation that is supported by a computationally efficient and easy-to-use algorithm.

Another approach based on multiscaling analysis on the energy equation with classical diffusion is provided in [17]. In this reference, the authors analyze a new equilibrium distribution function to overcome the observe inconsistencies of fluxes at the interface of a fluid and solid with different densities or specific heat capacities. Both mentioned references are of interest to describe recent relevant algorithms on fire and reaction problems where the energy equation is employed. Considering other areas of research, it shall be noted that the Darcy law involving nanofluids has been considered in [27]. The authors show the different solution profiles obtained after the numerical exercise where an exponential decay can be perceived. Such an exponential decay is immediate in the linear diffusion case, nonetheless for the non-linear diffusion further analytical assessments shall be done. In the same direction, [27] describes the effects of swimming gyrotactic microorganisms for magnetohydrodynamics nanofluid using Darcy law. In this case, the authors use linear differential equations with variable coefficients that precludes the existence of propagation principles characterized by the Chebyshev spectral collocation method. Further, in [23] the authors study a non-newtonian description of a nanofluid with nanoparticles volume motion via numerical assessment.

The effect of porosity in a partial slip for a peristaltic transport in a Jeffrey fluid has been investigated in [21]. In addition [7] employs a non-linear diffusion to improve the accuracy in simulating potential coagulation in an electromagnetic blood flow in annular vessel geometries. In all the cited cases, the finite speed of propagation defines a diffusive front. This is a common behaviour to all porous medium equations and shall be characterized for the problem discussed along the present work.

In relation with the techniques used along the presented analysis, it shall be mentioned that they are based on analytical approximations under the parabolicity condition in the non-linear diffusion. There are other positibilities to introduce the analytical conceptions based on dynamical systems. For instance, in [30], the author introduces the Raman soliton model for nanoscale optical waveguides in metamaterials to determine periodic, heteroclinic and homoclinic solutions.

2. Materials and methods

The methods followed along this research are based on an analytical approximation together with a representative testing for the validation of each of the physical laws and parameters involved.

The driving equation is based on a non-linear diffusion which is of relevance to predict accurately the behaviour of the propagating front inherent to any discharge process. The pressure induced by the discharge makes the suppressant to travel along the domain with finite propagation. Such finite propagation feature is precisely determined with a non-linear diffusion of the form [28]:

$$\begin{aligned} &u_t = \nabla \cdot (u^{m-1} \nabla u), \\ &m > 1, \end{aligned} \tag{2.1}$$

where u^{m-1} is referred as the pressure variable.

The proposed non-linear diffusion shall be justified further. For this purpose, a flight test has been performed on a Airbus A400M instrumentalized engine nacelle. Previously, it has been splited into several sections as per Figure 1.

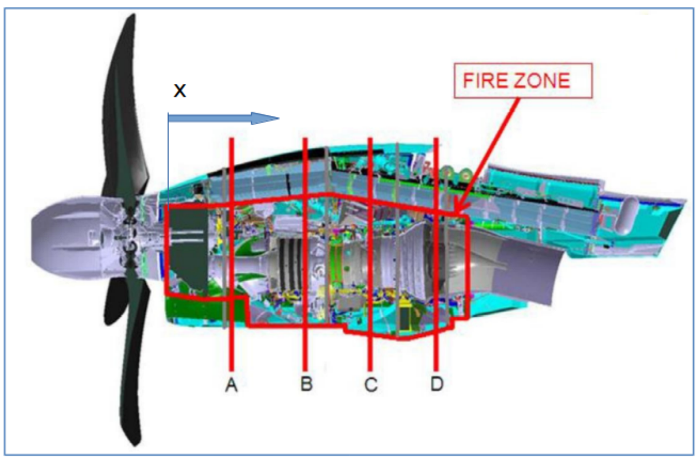

Figure 1. Domain representation. Note that the sensors are placed close to each of the sections A, B, C and D

For each section in Figure 1 a set of fire extinguisher sensors where located with the intention of measuring the agent concentration along the domain, see Figures 2 and 3.

The extinguishing system design point shall consider the most demanding scenario (i.e, the agent bottle at lowest temperature and highest dynamic pressure or ventilation in the engine cowl). Then the flight altitude was considered at the lowest level in cruise (150 ft) with the highest allowed airspeed (Mach 0.45) at ISA-70°C and a bottle temperature of -55°C. The flight test used to validate the diffusion approximation in (2.1) consisted on a procedure for the flight crew to push on the fire extinguishing button. Once the fire extinguisher was released, the sensors measure the agent concentration per volume as per Figure 4.

Note that once the discharge begins, the agent starts to travel and at the stagnation condition at the entrance of each sensor, a finite propagating diffusion acts.

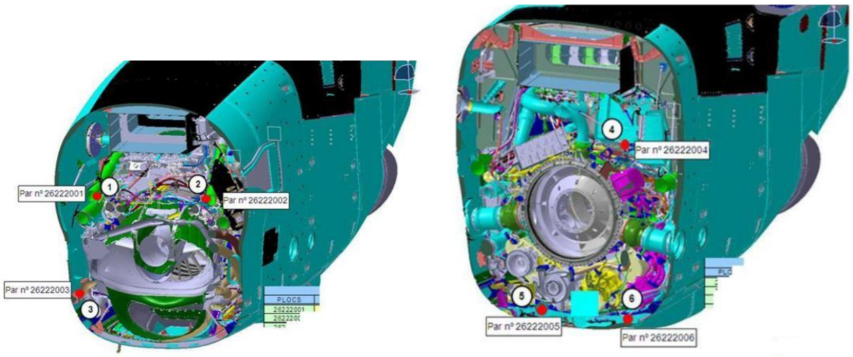

Figure 2. Extinguisher sensor position and numbering for Section A (left) and Section B (right).

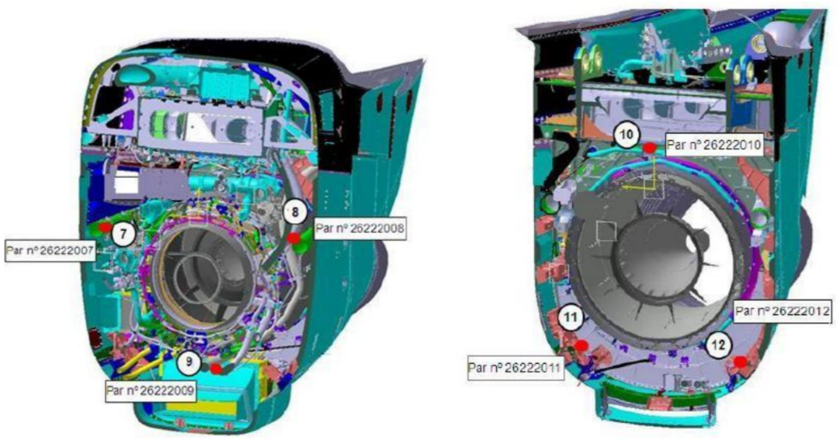

Figure 3. Extinguisher sensor position and numbering for Section C (left) and Section D (right).

Initially, this indicates that (2.1) may be an appropriate candidate to model the mass transfer procedure. A scaling analysis on (2.1) provides:

$$[u] \sim [D]^{\frac{2}{m-1}} \ [T]^{\frac{1}{m-1}}, \tag{2.2}$$

where D is a characteristic nacelle dimension (for instance the diameter) and T a characteristic time related with the agent evolution. Considering the time evolution, $[u] \sim [T]^{\frac{1}{m-1}}$. A particular relation between u and T can be obtained upon the data in Figure 4. For this purpose, the main increasing rate is given for an interval of 1.3 seconds. This interval is considered and a particular analytical mean function is determined:

$$[u] \sim [T]^3 \ \rightarrow \ m = \frac{4}{3}. \tag{2.3}$$

In addition, a non-linear advection ($c \cdot \nabla u^q$) is introduced to model the forced convection induced by the pressurized bottle at discharging. A postulated reaction term of the form $|x|^\sigma u^p$, $\sigma > 0$ and $p < 1$ aims to account for the natural heterogeneity involved in all fire suppresant process. Note that the adequacy of each of the involved terms (diffusion, advection and reaction) to model the fire extinguishing process is justified in Section 3.4. In the same section, we determine the different

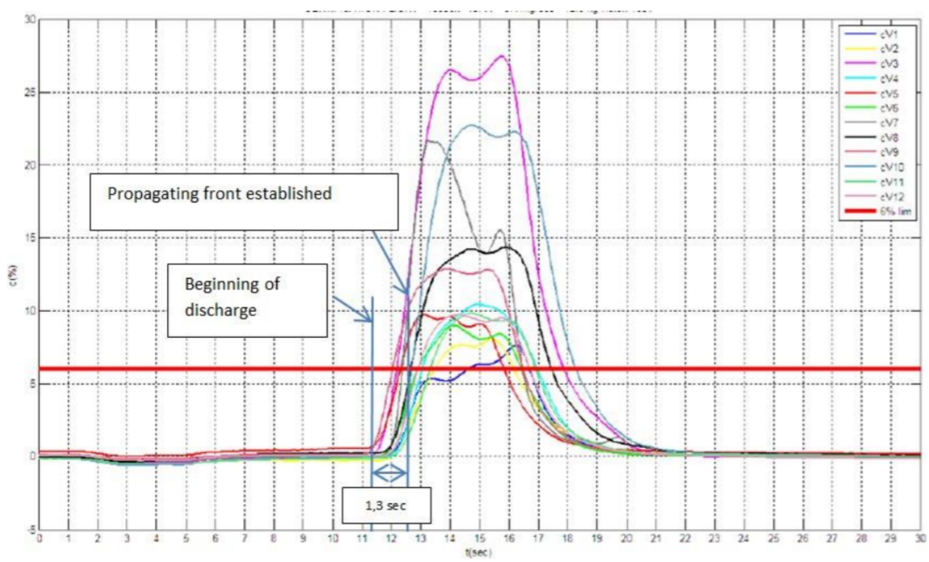

Figure 4. Extinguisher volumetric concentration evolution upon fire activation procedure that is initiated during the flight test used to validate the postulated diffusion. Each of the coloured lines represent the measurements of a sensor located as described in the previous figures. Note that the horizontal red bar represents a volumetric concentration of 6%. At 1.3 seconds from the beginning of discharge, half of the sensors reach the 6% level.

time variables to build a physical sound of the extinguishing process. Particularly: the time required for the suppressant to reach the whole domain including dead areas outside the forced convection induced by the discharge energy.

3. Discussion and Results

Any formulated modelling equation shall be shown to exhibit certain mathematical properties to ensure the existence and regularity of solutions. In addition and along this chapter, the different solutions are determined and the application exercise described.

The equation (Problem P) analyzed along the present work reads:

$$\begin{aligned} &u_t = \Delta u^m + |c \cdot \nabla u^q| + |x|^\sigma u^p, \\ &u\,(x,0) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \\ &m > 1,\, q \geq 1\,, \sigma > 0\,, 0 < p < 1. \end{aligned} \tag{3.1}$$

As described, the non-linear diffusion permits to account for certain mathematical properties of interest such us the finite propagating support. The non-linear advection aims to consider the extinguisher induced energy and movement while the reaction term accounts for an heterogeneous process. A fire extinguishing process starts with the discharge of a fire suppressant agent that propagates along the domain (in this case, the red zone in Figure 1). Such propagation is hypothesized to be characterized by the Problem P based on the following rationale:

- The non-linear diffusion aims to model the finite speed of agent propagation along the domain as measured in Figure 4. Once the discharge occurs, a diffusion interface exists that segregates between zones with agent ($u > 0$) and zones without agent ($u = 0$). The introduction of such interface is relevant to consider complex geometries, as it is the case of the aircraft engine nacelle. In such nacelles, there might by certain zones, out of the advection influence, where the non-linear diffusion is predominant (zones known as dead zones).
- The non-linear advection represents all kind of forced propagation. Once the bottles activates, the pressurized fire suppressant starts to populate the engine nacelle. Such process releases an energy that was previously confined within the bottle and that is considered to induce a force convection.
- The reaction term functional behaviour introduces the natural saturation in the agent evolution and the heterogeneous distribution along the domain. The extinguisher propagation process is fast at the beginning as there is not suppressant in the engine nacelle. Once the discharge occurs and the suppressor evolves, the agent increases while reducing the time rate (u_t). This principle is introduced by the term u^p ($p < 1$). In addition, the agent distribution along the domain is heterogeneous. The discharge nozzles are placed in different positions, so that the time rate varies with the position, and such variation is according to the law $|x|^\sigma$ ($\sigma > 1$).

3.1. Existence and uniqueness

Firstly, consider the following definitions

Definition 1. Consider $f \in C^{2+\gamma,\,1+\gamma/2}(\mathbb{R}^d \times (0,T))$, $f^m \in C^{2+\gamma}(\mathbb{R}^d \times (0,T))$ for $\gamma > 0$ such that the operator L is defined as:

$$Lf = f_t - \Delta f^m - c \cdot \nabla f. \tag{3.2}$$

Note that the diffusivity in P is $D(u) = u^{m-1}$, which is positive provided $u > 0$ globally or locally in the classical ball of radium R ($B_R \in \mathbb{R}^d$). This positivity condition prevents the degeneracy due to the non-linear diffusion in u. The previous works in [3,5,6,19,29] and [14] proved the existence of solutions for an aggregation equation with no reaction-absorption term. Nonetheless, the lack of a Lipschitz condition in P makes the equation in u to loss the regularity introduced by the positive diffusivity. Consequently, our aim is to show the existence of solutions by the use of the cited references and considering, in addition, the specific existence questions introduced by the non-Lipschitz reaction term.

The interior regularity of the parabolic operator permits to consider the following definition:

Definition 3.1. Consider two possible solutions $\hat{u}$ and $\tilde{u} \in C^{2+\gamma,\,1+\gamma/2}(\mathbb{R}^d \times (0,T))$. The function $\hat{u}$ is an upper solutions if:

$$L\hat{u} \geq |x|^\sigma \hat{u}^p. \tag{3.3}$$

The function $\tilde{u}$ is a lower solutions if:

$$L\tilde{u} \leq |x|^\sigma \tilde{u}^p. \tag{3.4}$$

Note that:

$$\tilde{u}(x,0) \leq u(x,0) \leq \hat{u}(x,0), \tag{3.5}$$

where

$$\hat{u}(x,0) = u(x,0) + \nu,\ \tilde{u}(x,0) = u(x,0) - \nu, \tag{3.6}$$

being

$$0 < \nu < \min_{x\in\mathbb{R}^d}\{u(x,0)\}. \tag{3.7}$$

We define the monotone sequence $\{u^{(k)}\}_{k=0,1,2,...}$ with $u^{(0)} = u_0(x)$, such that:

$$\begin{aligned} &u_t^{(k)} = \Delta(u^{(k)})^m + c\cdot\nabla(u^{(k)})^q + |x|^\sigma(u^{(k-1)})^p,\\ &u^{(0)}(x,0) = u_0(x),\ k = 1,2,3\ldots \end{aligned} \tag{3.8}$$

With the intention to prove the monotone and ordered behaviour of $\{u^{(k)}\}$, the recurrent approach in Ch. 12 of [24] shall be followed. In addition, the existence of solutions for each $u^{(k)}$ is ensured based on the fact that the operator L is positive for a positive f (refer to Ch. 3 [28] and Theorems 2.1 and 2.2, Ch. 7 [24]). Therefore:

$$\tilde{u} \leq \tilde{u}^{(k+1)} \leq \hat{u}^{(k+1)} \leq \hat{u}, \tag{3.9}$$

and ordered:

$$\hat{u}^{(k)} \geq \hat{u}^{(k+1)};\ \tilde{u}^{(k)} \leq \tilde{u}^{(k+1)}. \tag{3.10}$$

Note that $\{\hat{u}^{(k)}\}$ is non-increasing while $\{\tilde{u}^{(k)}\}$ is non-decreasing. Such monotone and ordered features permits to postulate the following existence theorem:

Theorem 3.1. *Assume $u \geq \epsilon > 0$ in the ball $B_R \times (0,T)$, then, the sequences $\{\hat{u}^{(k)}\}$ and $\{\tilde{u}^{(k)}\} \in C^{2+\gamma,\,1+\gamma/2}(B_R \times (0,T))$ are convergent.*

Proof. Consider the test function $\phi \in C^\infty(B_R)$, such that for $0 < \tau < t < T$, the weak upper and lower solutions $\hat{u}^{(k)}$ and $\tilde{u}^{(k)}$ read:

$$\begin{aligned} \int_{B_R} \hat{u}^{(k)}(t)\,\phi(t) = &\int_{B_R} (u_0 + \nu(k))\,\phi(0)\\ &+ \int_0^t \int_{B_R} [\hat{u}^{(k)}\,\phi_t + \hat{u}^{(k)\,m}\,\Delta\phi + c\cdot\nabla\phi\,\hat{u}^{(k)q} + |x|^\sigma\hat{u}^{(k)p}\,\phi]ds, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \int_{B_R} \tilde{u}^{(k)}(t)\,\phi(t) = &\int_{B_R} (u_0 - \nu(k))\,\phi(0)\\ &+ \int_0^t \int_{B_R} [\tilde{u}^{(k)}\,\phi_t + \tilde{u}^{(k)\,m}\,\Delta\phi + c\cdot\nabla\phi\,\tilde{u}^{(k)q} + |x|^\sigma\tilde{u}^{(k)p}\,\phi]ds, \end{aligned} \tag{3.12}$$

where $\lim_{k\to\infty}\nu = 0$. Now, we make use of the dominated convergence theorem to have:

$$\begin{aligned}&\int_{B_R} \lim_{k\to\infty} \hat{u}^{(k)}(t)\,\phi(t)\\ &= \int_{B_R} (u_0 + \lim_{k\to\infty} \nu(k))\,\phi(0)\\ &+ \int_0^t \int_{B_R} [\lim_{k\to\infty} \hat{u}^{(k)}\,\phi_t + \lim_{k\to\infty} \hat{u}^{(k)\,m}\,\Delta\phi + c\cdot\nabla\phi \lim_{k\to\infty} \hat{u}^{(k)q} + |x|^\sigma \lim_{k\to\infty} \hat{u}^{(k)p}\,\phi]ds,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\int_{B_R} \lim_{k\to\infty} \tilde{u}^{(k)}(t)\,\phi(t)\\ &= \int_{B_R} (u_0 - \lim_{k\to\infty} \nu(k))\,\phi(0)\\ &+ \int_0^t \int_{B_R} [\lim_{k\to\infty} \tilde{u}^{(k)}\,\phi_t + \lim_{k\to\infty} \tilde{u}^{(k)\,m}\,\Delta\phi + c\cdot\nabla\phi \lim_{k\to\infty} \tilde{u}^{(k)q} + |x|^\sigma \lim_{k\to\infty} \tilde{u}^{(k)p}\,\phi]ds,\end{aligned} \tag{3.14}$$

after substraction:

$$\begin{aligned}&\int_{B_R} \lim_{k\to\infty} (\hat{u}^{(k)} - \tilde{u}^{(k)})(t)\,\phi(t)\\ &= \int_0^t \int_{B_R} [\lim_{k\to\infty} (\hat{u}^{(k)} - \tilde{u}^{(k)})\,\phi_t + \lim_{k\to\infty} (\hat{u}^{(k)\,m} - \tilde{u}^{(k)\,m})\,\Delta\phi\\ &+ c\cdot\nabla\phi \lim_{k\to\infty} (\hat{u}^{(k)q} - \tilde{u}^{(k)q}) + |x|^\sigma \lim_{k\to\infty} (\hat{u}^{(k)p} - \tilde{u}^{(k)p})\,\phi]ds.\end{aligned} \tag{3.15}$$

Consider $M = \max\{|x|^\sigma\}_{B_R}$ together with the condition $u \geq \epsilon > 0$:

$$\begin{aligned}&\int_{B_R} \lim_{k\to\infty} (\hat{u}^{(k)} - \tilde{u}^{(k)})(t)\,\phi(t)\\ &\leq \int_0^t \int_{B_R} [\lim_{k\to\infty} (\hat{u}^{(k)} - \tilde{u}^{(k)})\,\phi_t + \lim_{k\to\infty} (\hat{u}^{(k)m-1}(\hat{u}^{(k)} - \tilde{u}^{(k)}))\,\Delta\phi\\ &+ c\cdot\nabla\phi \lim_{k\to\infty} (\hat{u}^{(k)q-1}(\hat{u}^{(k)} - \tilde{u}^{(k)})) + M \lim_{k\to\infty} \left(\frac{1}{\tilde{u}^{(k)1-p}} (\hat{u}^{(k)} - \tilde{u}^{(k)})\right)\phi]ds\end{aligned} \tag{3.16}$$

the last inequality holds when $\hat{u}^{(k)} \to \tilde{u}^{(k)}$ on the right hand side, so that we conclude on the convergence of the left hand side integral. Additionally, the regular convergence in the upper and lower positive solutions permits to state that such solutions satisfy the regularity embedding:

$$C^{2+\gamma,\,1+\gamma/2}(B_R \times (0,T)) \subset L^1(\,(0,T); W^{2,1}(B_R\,) \cap W^{1,1}(\,(0,T); L^1(B_R)\,). \tag{3.17}$$

□

Theorem 3.2. *Assume $u \geq \epsilon > 0$ is a solution to P in $\mathbb{R}^d \times (0,T)$, then u is unique.*

Proof. Consider $\hat{u}$ is a maximal solution to problem P in $\mathbb{R}^d \times (0,T)$ such that:

$$\hat{u}(x,0) = u_0(x), \tag{3.18}$$

defined as:

$$\hat{u}_t = \Delta \hat{u}^m + c \cdot \nabla \hat{u}^q + |x|^\sigma \hat{u}^p. \tag{3.19}$$

In addition, consider the minimal solution u:

$$\begin{aligned} u_t &= \Delta u^m + c \cdot \nabla u^q + |x|^\sigma u^p, \\ u(x,0) &= u_0(x). \end{aligned} \tag{3.20}$$

Then, upon definition of a test function $\phi \in C^\infty(\mathbb{R}^d)$, the following holds:

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d} (\hat{u} - u)(t)\, \phi(t) \\ &= \int_0^t \int_{\mathbb{R}^d} [(\hat{u} - u)\, \phi_t + (\hat{u}^m - u^m)\, \Delta\phi + c \cdot \nabla\phi\, (\hat{u}^q - u^q) + |x|^\sigma (\hat{u}^p - u^p)\, \phi] ds. \end{aligned} \tag{3.21}$$

Now, define:

$$a_1(x,s) = \begin{bmatrix} \frac{\hat{u}^m - u^m}{\hat{u} - u} \text{ for } \hat{u} \not\equiv u \\ m u^{m-1} \text{ otherwise} \end{bmatrix}, \tag{3.22}$$

such that for two fixed values of x and $s = t \leq T$:

$$0 \leq a_1(x,s) \leq c_1(m, \|u_0\|_\infty, T). \tag{3.23}$$

Consider the particular test function:

$$\phi(|x|, s) = e^{k(T-s)} (1 + |x|^2)^{-\gamma}, \tag{3.24}$$

for some constants k and γ.

The test function verifies:

$$\phi_t = -k\phi(x,s), \;\; |\nabla_{|x|}\phi| \leq c_3(\gamma, d)\phi(x,s), \;\;\; \Delta_{|x|}\phi \leq c_4(\gamma, d)\phi(x,s), \tag{3.25}$$

such that:

$$\begin{aligned} &(\hat{u} - u)\, \phi_t + (\hat{u}^m - u^m)\, \Delta\phi + c \cdot \nabla\phi\, (\hat{u}^q - u^q) + |x|^\sigma (\hat{u}^p - u^p)\, \phi \\ &\leq -(\hat{u} - u)\, k\, \phi + a_1 (\hat{u} - u)\, c_4 \phi + c\, c_3 \phi\, a_1 (\hat{u} - u) + L_1 (\hat{u} - u)\, \phi, \end{aligned} \tag{3.26}$$

where L_1 is the Lipschitz constant.

The constant k shall be selected such that:

$$-k + a_1 c_4 + a_1 c_3 c \leq 0. \tag{3.27}$$

It suffices to consider:

$$k \geq a_1 (c_4 + c_3 c), \tag{3.28}$$

so that:

$$(-k + a_1 c_4 + a_1 c_3 c)\phi(\hat{u} - u) \leq 0. \tag{3.29}$$

The expressions (3.21) reads:

$$0 \leq \int_{\mathbb{R}^d} (\hat{u} - u)(t)\,\phi(t) \leq \int_0^t \int_{\mathbb{R}^d} L_1(\hat{u} - u)\,\phi ds. \tag{3.30}$$

Making the d/dt, considering the smoothness of the test function and the Hölder continuity of solutions in (3.17) the following holds:

$$0 \leq \frac{d}{dt} \int_{\mathbb{R}^d} (\hat{u} - u)(t)\,\phi(t) \leq \int_{\mathbb{R}^d} L_1(\hat{u} - u)(t)\,\phi(t). \tag{3.31}$$

This can be shown considering the following:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{d}{dt}\left((\hat{u} - u)(t)\,\phi(t)\right) &= \int_{\mathbb{R}^d} \left(\frac{d}{dt}(\hat{u} - u)(t) - k(\hat{u} - u)(t)\right)\,\phi(t) \\ &\leq \int_{\mathbb{R}^d} \frac{d}{dt}(\hat{u} - u)(t)\,\phi(t). \end{aligned} \tag{3.32}$$

Given the Hölder continuity in the first time derivative, it is possible to find a suitable constant (admit this is L_1) so that:

$$\int_{\mathbb{R}^d} \frac{d}{dt}(\hat{u} - u)(t)\,\phi(t) \leq \int_{\mathbb{R}^d} L_1(\hat{u} - u)(t)\,\phi(t). \tag{3.33}$$

Now, define:

$$g(t) = \int_{\mathbb{R}^d} (\hat{u} - u)(t)\,\phi(t), \tag{3.34}$$

then

$$\frac{dg(t)}{dt} \leq L_1 g(t), \tag{3.35}$$

with null initial condition $g(0) = 0$. The equation (3.35) is verified by the null condition $g(t) = 0$, then $\hat{u} = u$. Similarly and returning to (3.30):

$$0 \leq \int_{\mathbb{R}^d} (\hat{u} - u)(t)\,\phi(t) \leq 0, \tag{3.36}$$

to conclude:

$$\hat{u} = u, \tag{3.37}$$

showing, then, the uniqueness of any positive solution to P. □

3.2. Main estimates and blow-up existence

Consider the following initial condition with $R \in (0, \infty)$ and $\epsilon > 0$ arbitrary small:

$$u_0 = 0 \;\; in \;\; B_R = \{R - \epsilon < |x - x_0| < R + \epsilon\}. \tag{3.38}$$

The non-Lipschitz reaction induces non-uniqueness. Then, there exist two enveloping solutions: A trivial subsolution $\tilde{u} = 0$ and a supersolution of the general form:

$$\hat{u} = K|x|^{\gamma}(t - \tau)^{\alpha}, \tag{3.39}$$

where $\tau > 0$, $\gamma > 0$ and $K > 0$ to be determined. Upon substitution into P:

$$|x|^{\gamma} K\alpha(t-\tau)^{\alpha-1} = m\gamma(m\gamma-1)|x|^{m\gamma-2}K^m(t-\tau)^{m\alpha} + cK^q q\gamma|x|^{q\gamma-1}(t-\tau)^{q\alpha} + |x|^{p\gamma+\sigma}K^p(t-\tau)^{p\alpha}. \tag{3.40}$$

Note that the slow diffusion ($m > 1$) makes $u = 0$ in compact subsets of B_R, therefore in such compact subsets the advection and reaction are considered to be predominant over the diffusion. After an elementary calculation making $q\gamma - 1 = p\gamma + \sigma$:

$$\gamma = \frac{\sigma+1}{q-p}. \tag{3.41}$$

A positive value for α is obtained making $\alpha - 1 = \alpha p$:

$$\alpha = \frac{1}{1-p}. \tag{3.42}$$

Now, K is expressed as $K = K_1 + K_2$ where K_1 is obtained considering the reaction only and K_2 the advection term separately:

$$K_1 = (1-p)^{\frac{1}{1-p}}, \quad K_2 = \left(\frac{q-p}{q(1-p)(\sigma+1)}\right)^{\frac{1}{q-1}}. \tag{3.43}$$

As a consequence:

$$\hat{u}_{\tau\to 0} = \left((1-p)^{\frac{1}{1-p}} + \left(\frac{q-p}{q(1-p)(\sigma+1)}\right)^{\frac{1}{q-1}}\right)|x|^{\frac{\sigma+1}{q-p}}(t)^{1/(1-p)}. \tag{3.44}$$

The maximal solution (3.44) is valid provided that the advection and reaction are predominat over the diffusion:

$$q\gamma - 1 > m\gamma - 2, \tag{3.45}$$

and considering the expression (3.41), we have:

$$\frac{\sigma+1}{\sigma+2} < \frac{q-p}{m-p}. \tag{3.46}$$

This last condition can be used to distinguish between the following results depending on the data parameters for P:

- $\frac{\sigma+1}{\sigma+2} < \frac{q-p}{m-p}$.

  The advection and reaction terms are predominant. The non-Lipschitz condition in the reaction induces non-uniqueness and a supersolution $\hat{u}$ and a subsolution $\tilde{u} = 0$ have been obtained for initial data as per (3.38).

- $\frac{\sigma+1}{\sigma+2} \geq \frac{q-p}{m-p}$ or $m \geq \left(\frac{\sigma+2}{\sigma+1}\right)(q-p) + p$.

  The diffusion is predominant and we can expect a finite propagation in compact subsets of B_R. A solution for this case is precisely assessed in Theorem 3.5.

The next intention is to provide a condition to distinguish between the existence of a global solution and the explosion in finite time with the definition of a critical exponent p^*.

Theorem 3.3. *Consider the critical exponent* $0 \leq p^* < 1$ *defined as:*

$$p^* = \inf\{\left(q - \frac{1}{4}(\sigma+1)(m-1)\right)_+, 1\}. \tag{3.47}$$

For:

$$p > p^*, \tag{3.48}$$

blow up or explosion in finite time exists, while for:

$$p \leq p^*, \tag{3.49}$$

a global solution exists.

Proof. Consider the self-similar profile:

$$G(x,t) = t^{-\alpha} f(|x|t^{\beta}), \;\; \xi = |x|t^{\beta}. \tag{3.50}$$

We make $d = 1$ in the sake of simplicity. Upon substitution into P:

$$-\alpha t^{-\alpha-1} f + \beta \underbrace{|x|t^{\beta}}_{\xi} t^{-\alpha-1} f' = t^{-\alpha m} t^{2\beta} f^m_{xx} + c t^{-\alpha q} t^{\beta} f^q_x + \xi^{\sigma} t^{-\sigma\beta-\alpha p} f^p. \tag{3.51}$$

Making the following equalities in (3.51):

$$-\alpha - 1 = -\alpha m + 2\beta, \quad -\alpha q + \beta = -\beta\sigma - \alpha p. \tag{3.52}$$

So that:

$$\alpha = \frac{\sigma+1}{2\left((p-q) + (\sigma+1)\left(\frac{m-1}{2}\right)\right)}, \quad \beta = q - p. \tag{3.53}$$

The existence of finite time blow up requires:

$$2\left((p-q) + (\sigma+1)\left(\tfrac{m-1}{2}\right)\right) > 0, \quad p > \left(q - \tfrac{1}{4}(\sigma+1)(m-1)\right). \tag{3.54}$$

So that, the critical exponent is defined as:

$$p^* = \left(q - \frac{1}{4}(\sigma+1)(m-1)\right)_+, \tag{3.55}$$

and blow up exists if $p > p^*$ while global solutions exists if $p \leq p^*$.

Note that $0 < p^* < 1$ in accordance with the data condition $0 < p < 1$. In addition, $p^* = 0$ corresponds to the case of:

$$\left(q - \frac{1}{4}(\sigma+1)(m-1)\right) \leq 0 \;\rightarrow\; \left(q - \frac{1}{4}(\sigma+1)(m-1)\right)_+ = 0. \tag{3.56}$$

Then, the appropriate interval for p^* is:

$$0 \leq p^* < 1, \tag{3.57}$$

which is achieved as per the expression:

$$p^* = \inf\{\left(q - \frac{1}{4}(\sigma+1)(m-1)\right)_+ ,1\} \tag{3.58}$$

□

Our next intention is to determine a precise estimation from below to account for the propagation features of the solutions to P:

Theorem 3.4. *Given $t_0 > 0$, the following propagation estimates hold:*

- $|t - t_0| << 1$
$$|x| \leq 2A^2 |\log(t-t_0)|^{\frac{3}{2}}, \tag{3.59}$$
- $|t - t_0| >> 1, \quad 1 - p \geq m - q$
$$|x| \leq \left(2\mu A^{\delta}\right)^{\frac{1}{1-\sigma}} \; t^{\frac{2(1-\delta)+3}{2(1-\sigma)}}, \tag{3.60}$$
- $|t - t_0| >> 1, \quad 1 - p < m - q$
$$|x| \leq 2\lambda c A^{\frac{q-1}{m-1}+1} t^{\frac{5m-(3+2q)}{2(m-1)}}, \tag{3.61}$$

where μ, δ, λ are constants depending on the data parameters q, m and p. A is chosen such that the propagation results from a subsolution to P:

$$0 < A < t_0 \max_{x \in \mathbb{R}} \{v_0(x)\} + |x| |\log t_0|^{-\frac{1}{2}}. \tag{3.62}$$

Proof. Firstly, the pressure variable v reads:

$$v = \frac{m}{m-1} u^{m-1}, \tag{3.63}$$

such that the problem P adopts the form:

$$v_t = (m-1)v\Delta v + |\nabla v|^2 + \lambda v^{\theta} |c \cdot \nabla v| + \mu |x|^{\sigma} v^{\delta}, \tag{3.64}$$

where

$$\begin{aligned} \lambda &= \tfrac{q}{m-1}, \quad \theta = \tfrac{q-1}{m-1}, \\ \delta &= \tfrac{p+m-2}{m-1}, \quad \mu = m\left(\tfrac{m-1}{m}\right)^{\delta}. \end{aligned} \tag{3.65}$$

Consider the following propagating wave function (refer to [19]):

$$w(x,t) = \frac{1}{t}\left(A - |x|(\log(t-t_0))^{-\frac{1}{2}}\right)_+ \tag{3.66}$$

where $A > 0$ and $t_0 \geq 0$ are selected so that $w(x,t)$ is a subsolution to P. We make the calculation assuming $t_0 = 0$. Then:

$$\begin{aligned} &-\frac{1}{t^2}\left(A - |x|(\log t)^{-\frac{1}{2}}\right) + \frac{1}{t}\left(A + |x|\frac{1}{2t}(\log t)^{-\frac{3}{2}}\right) \\ \leq &\frac{1}{t^2}\left(A - (\log t)^{-\frac{1}{2}}\right)^2 + \lambda t^{-\frac{q-1}{m-1}}\left(A - |x|(\log t)^{-\frac{1}{2}}\right)^{\frac{q-1}{m-1}} \frac{c}{t}\left(A - (\log t)^{-\frac{1}{2}}\right) \\ &+ \mu |x|^{\sigma} t^{-\delta}\left(A - |x|(\log t)^{-\frac{1}{2}}\right)^{\delta}. \end{aligned} \tag{3.67}$$

The following inequality holds provided $\log t > 0$. In case not, it suffices to consider a $t_0 \geq 0$ such that $\log(t - t_0) > 0$. Note that, we continue operating with $\log t > 0$ for the sake of simplicity:

$$|x|\frac{1}{2t}(\log t)^{-\frac{3}{2}} \leq A+|x|\frac{1}{2t}(\log t)^{-\frac{3}{2}} \leq \frac{1}{t}A^2+\lambda c t^{-\frac{q-1}{m-1}}A^{\frac{q-1}{m-1}+1}+\mu|x|^{\sigma}t^{-\delta}A^{\delta}. \quad (3.68)$$

Upon recovery of the t_0 translation, we have:

$$\begin{aligned}|x| \leq & 2A^2(\log(t-t_0))^{\frac{3}{2}} + 2\lambda c(t-t_0)^{\frac{m-q}{m-1}}A^{\frac{q-1}{m-1}+1}(\log(t-t_0))^{\frac{3}{2}} \\ & + 2\mu|x|^{\sigma}(t-t_0)^{1-\delta}A^{\delta}(\log(t-t_0))^{\frac{3}{2}}.\end{aligned} \quad (3.69)$$

We consider the first term on the right hand side to only account for the log profile. This means:

$$|x| \leq 2A^2|\log(t-t_0)|^{\frac{3}{2}}. \quad (3.70)$$

In addition, we assess the case $t >> t_0$ under the conditions $1-p \geq m-q$ and $1-p < m-q$. For the case $1-p \geq m-q$, we have :

$$|x| \leq \left(2\mu t^{1-\delta}A^{\delta}(\log t)^{\frac{3}{2}}\right)^{\frac{1}{1-\sigma}} \leq \left(2\mu A^{\delta}\right)^{\frac{1}{1-\sigma}}\ t^{\frac{2(1-\delta)+3}{2(1-\sigma)}}. \quad (3.71)$$

In the case of $1-p < m-q$:

$$|x| \leq 2\lambda c t^{\frac{m-q}{m-1}}A^{\frac{q-1}{m-1}+1}(\log t)^{\frac{3}{2}} \leq 2\lambda c A^{\frac{q-1}{m-1}+1}t^{\frac{5m-(3+2q)}{2(m-1)}}. \quad (3.72)$$

Note that the w profile (3.66) is a subsolution, then an appropriate value for A is obtained as:

$$v(x,t_0) = v_0(x) \geq \frac{1}{t_0}\left(A - |x|(\log(t_0))^{-\frac{1}{2}}\right)_+. \quad (3.73)$$

It suffices to consider:

$$0 < A < t_0 \max_{x\in\mathbb{R}}\{v_0(x)\} + |x||\log t_0|^{-\frac{1}{2}}, \quad (3.74)$$

which concludes the proof. □

Our next objective is to provide a precise subsolution for u in the proximity of the propagating support. For this purpose we consider a selfsimilar solution (3.50) so that the following theorem holds.

Theorem 3.5. *Consider the selfsimilar solution* (3.50) *with* $t \gg 1$, *we search for a bell shape solution* f *satisfying:*

$$f(\xi) > 0, \quad for \quad 0 < \xi \leq \xi_0, \quad and \quad (f^q)'(\xi_0) = 0. \quad (3.75)$$

$$(f^q)'(\Theta) \ll 1, \quad for \quad \Theta \in (\xi_0 - \epsilon, \xi_0), \quad (3.76)$$

for ϵ *arbitrary small. Then, the following estimation holds:*

$$f(\xi) \geq c(\alpha,\beta,d,m,p)^{\frac{1}{m-1}}|x|^{\frac{\sigma}{1-p}}, \quad c(\alpha,\beta,d,m,p) = (-\alpha+\beta d)^{\frac{m-1}{p-1}}, \quad (3.77)$$

for $0 < \xi \leq \xi_0$.

In addition, the support ξ_0 *is obtained upon distinction of two cases depending on the relative strength for the advection and diffusion:*

- $q \geq m-1$. *The evolving support is:*

$$\xi_0 = c_{supp}|x|^{\frac{\sigma(m-1)}{2(1-p)}}, \quad c_{supp} = \frac{(-\alpha+\beta d)^{\frac{m-1}{2(p-1)}}}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}}. \tag{3.78}$$

- $q < m-1$. *The support is* $\mathbb{R}^d$*:*

$$\xi_0 \to \infty. \tag{3.79}$$

Proof. Consider the following problem:

$$u_t = \Delta u^m + |c \cdot \nabla u^q| + |x|^\sigma u^p \geq \Delta u^m + |c \cdot \nabla u^q| + H_{\gamma,n}, \tag{3.80}$$

where

$$H_{\gamma,n} = n^\sigma \min\{u^p, \gamma^{p-1}u\}, \quad n > 0 \text{ and } \gamma > 0, \tag{3.81}$$

is a lower estimation to the reaction term that is recovered in the limit with $n \to \infty$ and $\gamma \to 0$. Therefore, the following problem (P_S) provides a subsolution to P:

$$u_t = \Delta u^m + |c \cdot \nabla u^q| + H_{\gamma,n}. \tag{3.82}$$

The function $H_{\gamma,n}$ satisfies the Lipschitz condition. The existence and uniqueness of solutions can be shown by a standard approximation on expanding domains (refer to [2, 14, 18, 19] for further assessments).

The following equation permits to assess a precise solution for the evolving tail $f(|x|t^\beta)$ in (3.50) for $\xi \in (\Theta, \xi_0)$:

$$-\alpha t^{-\alpha-1} f + \beta \xi t^{-\alpha-1} f' = t^{-\alpha m}(f^m)'' + \frac{d-1}{\xi}(f^m)' + H_{\gamma,n}, \tag{3.83}$$

where:

$$H_{\gamma,n}(f,t) = n^\sigma \min\{f^p, \gamma^{p-1} t^{\alpha(p-1)} f\}, \tag{3.84}$$

so that

$$H_{\gamma,n}(f,t) = n^\sigma \min\{t^{-\alpha p} f^p, \gamma^{p-1} t^{-\alpha} f\}. \tag{3.85}$$

Now, consider t (to be assessed) such that:

$$H_{\gamma,n}(f,t) \geq n^\sigma \, c_1 \, f \tag{3.86}$$

and c_1 can be chosen as $c_1 = n^{-\sigma}(-\alpha + \beta d)$ for the sake of simplicity during the resolution of (3.83).

A subsolution f to the bell shape tail is given by:

$$\beta \xi f' = (f^m)'' + \frac{d-1}{\xi}(f^m)' + \beta d f. \tag{3.87}$$

This equation is of elliptic type with solution [9]:

$$f(\xi) = (A - B\xi^2)^{\frac{1}{m-1}}, \quad A > 0, \quad B = \frac{(m-1)\beta}{2m}, \tag{3.88}$$

which is valid for a sufficiently large time to hold the inequality (3.86):

$$n^\sigma \min\{f^p, \gamma^{p-1} t^{\alpha(p-1)} f\} \geq n^\sigma \, c_1 \, f \;\to\; \min\{f^p, \gamma^{p-1} t^{\alpha(p-1)} f\} \geq c_1 \, f. \tag{3.89}$$

For the sublinear evolution:

$$\gamma^{p-1}t^{\alpha(p-1)} \geq n^{-\sigma}(-\alpha+\beta d). \tag{3.90}$$

Consider the relation $n=\frac{1}{\gamma}$ to assess the effect of γ and n^σ.

An explicit value for t_γ is obtained from (3.90):

$$t_\gamma = (-\alpha+\beta d)^{\frac{-1}{\alpha(1-p)}}\gamma^{-1/\alpha}\left(\frac{1}{\gamma}\right)^{\frac{\sigma}{\alpha(1-p)}}. \tag{3.91}$$

The expression (3.88) is a subsolution for $t \geq t_\gamma$.

Any solution to P_S is indeed a subsolution to P as $H_{\gamma,n} \leq n^\sigma u^p$. To show this, consider w to be a solution to P and u a solution to P_S at $t=t_\gamma$. For any $\tau > t_\gamma$:

$$w(x,\tau) \geq u(x,t_\gamma), \tag{3.92}$$

for $t \geq 0$:

$$w(x,\tau+t) \geq u(x,t_\gamma+t). \tag{3.93}$$

In the limit with $\tau \to 0$:

$$w(x,t) \geq u(x,t), \quad x \in \mathbb{R}. \tag{3.94}$$

Showing that $u(x,t)$, solution to P_S, is indeed a subsolution to the problem P.

Now, note that our intention is to determine the evolution of the tail in the proximity of the support. For this purpose, consider that such tail is the asymptotic evolution departing from $\xi=0$ in the expression (3.88). This approach is required to determine a value for A:

$$f(\xi) = A^{\frac{1}{m-1}}, \tag{3.95}$$

Such value for A can be determined from (3.89) making $\xi=0$:

$$\min\{A^{\frac{p-1}{m-1}}, \gamma^{p-1}t^{\alpha(p-1)}\} \geq n^{-\sigma}(-\alpha+\beta d). \tag{3.96}$$

So that, in the minimal approximation:

$$A^{\frac{p-1}{m-1}} = n^{-\sigma}(-\alpha+\beta d), \ \rightarrow \ A = n^{\frac{\sigma(m-1)}{1-p}}(-\alpha+\beta d)^{\frac{m-1}{p-1}} = c_1(\alpha,\beta,d,m,p)n^{\frac{\sigma(m-1)}{1-p}}. \tag{3.97}$$

Considering the spatial variable $|x|$:

$$A(x) = c_1(\alpha,\beta,d,m,p)|x|^{\frac{\sigma(m-1)}{1-p}}. \tag{3.98}$$

So that, the evolution of the maximals in f is given as:

$$f(\xi) = c_1(\alpha,\beta,d,m,p)^{\frac{1}{m-1}}|x|^{\frac{\sigma}{1-p}}. \tag{3.99}$$

The searched function f satisfies: $(f^q)'(\xi_0)=0$. As a consequence, two cases shall be distinguish upon comparison of the parameters related with the advection and diffusion:

- $q \geq m-1$. In this case, the evolving support is determined as:

$$\begin{aligned} &f(\xi) = 0 \Rightarrow \xi = \left(\frac{A}{B}\right)^{(1/2)}, \\ &\xi_0 = \frac{1}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}} c_1^{(1/2)}(\alpha,\beta,d,m,p)|x|^{\frac{\sigma(m-1)}{2(1-p)}} = c_{supp}|x|^{\frac{\sigma(m-1)}{2(1-p)}}, \\ &c_{supp} = \frac{(-\alpha+\beta d)^{\frac{m-1}{2(p-1)}}}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}}. \end{aligned} \tag{3.100}$$

- $q < m-1$. The support is $\mathbb{R}^N$, i.e. $\xi_0 \to \infty$. □

As a consequence of the previous theorem, given a local $x = x_0$, any solution to P satisfies the following estimate:

$$u(x_0,t) \geq t^{-\alpha} c_1(\alpha,\beta,d,m,p)^{\frac{1}{m-1}} |x_0|^{\frac{\sigma}{1-p}}. \tag{3.101}$$

And in relation with the support:

- $q \geq m-1$:
$$|x - x_0| < c_{supp}|x|^{\frac{\sigma(m-1)}{2(1-p)}} t^{\beta}, \tag{3.102}$$
- $q < m-1$. The support is not finite.

Returning to the expression (3.51), we can distinguish between two cases to assess the asymptotic behaviour of solutions.

- If $-\alpha m + 2\beta > -\alpha q + \beta$: In this case and when $t >> 1$, the resulting equation is as per (3.83) whose solution has been obtained in Theorem 3.5.
- If $-\alpha m + 2\beta < -\alpha q + \beta$: The equation (3.51) adopts the following form at $t >> 1$:

$$-\alpha t^{-\alpha-1} f + \beta \underbrace{|x| t^{\beta}}_{\xi} t^{-\alpha-1} f' = c t^{-\alpha q} t^{\beta} f_x^q + \xi^{\sigma} t^{-\sigma\beta-\alpha p} f^p. \tag{3.103}$$

Considering this last equation, the following lemma holds:

Lemma 3.1. The self similar profile for (3.103) is of the form:

- If $q > 1$:
$$f(\xi) = \left(\frac{|\alpha|(q-1)}{cq}\right)^{\frac{1}{q-1}} \xi^{\frac{1}{q-1}}. \tag{3.104}$$
- If $q = 1$:
$$f(\xi) = (\beta\xi - c)^{\frac{\alpha}{\beta}}. \tag{3.105}$$

Proof. Assume the asymptotic separation of variables:

$$f(\xi) = e^{g(\xi)} + \psi(\xi) + \phi(\xi) + \ldots \quad \text{where} \quad e^{g(\xi)} \ll |\psi(\xi)| + |\phi(\xi)| \quad \text{as} \quad \xi \to \infty, \tag{3.106}$$

where $g'(\xi) > 0$. Considering the case $q > 1$ and the leading terms in (3.103), we have:

$$-\alpha = cqg'e^{(q-1)g} \;\to\; g(\xi) = \log\left(\frac{-\alpha(q-1)}{cq}\xi\right)^{\frac{1}{q-1}}. \tag{3.107}$$

Then, it suffices:

$$f(\xi) = \left(\frac{|\alpha|(q-1)}{cq}\right)^{\frac{1}{q-1}} \xi^{\frac{1}{q-1}}. \tag{3.108}$$

In case of $q = 1$ and considering again the leading terms in (3.103):

$$-\alpha = (c - \beta\xi)g' \;\to\; g(\xi) = \log(\beta\xi - c)^{\frac{\alpha}{\beta}}, \tag{3.109}$$

so that $f(\xi) = (\beta\xi - c)^{\frac{\alpha}{\beta}}$. □

3.3. Finite speed of propagation

The Porous Medium Equation, with or without advection, exhibits the well known property of finite speed of propagation whenever the solution tends to zero (as it is the case of compactly supported functions). Our objective is to characterize such finite speed of propagation and to determine a sharp estimation of the waiting time. The approximation follows some already available results in [9,10,14] and [12], although we introduce significant differences to account for a wider generalization. For this purpose, we make use of the pressure variable so that the resulting equation is as per the expression (3.64).

Theorem 3.6. *Consider $x_0 \in \mathbb{R}^d$, $T_0 > 0$, $D > 0$ and $\epsilon > 0$ with $\epsilon << 1$, such that the following function in the pressure variable:*

$$w(x,t) = D|x - x_0|^{\alpha}(T_0 - t)^{\beta}, \quad (\beta > 0), \tag{3.110}$$

is a local supersolution in $\Omega = \{|x - x_0| \le \epsilon\,,\;\; 0 \le t < T_0 < 1\}$. Where T_0 is the maximal waiting time to be determined.

Proof. We require $w(x,t)$ to be a local supersolution in ω, therefore:

$$w_t \le (m-1)w\Delta w + |\nabla w|^2 + \lambda w^{\theta}|c \cdot \nabla w| + \mu|x|^{\sigma}w^{\delta}, \tag{3.111}$$

where $\lambda, \theta, \mu, \delta$ have been defined in the set of expression (3.65). In addition

$$w(x,0) \le v_0(x), \quad |x - x_0| \le \epsilon \tag{3.112}$$

and

$$w(x_0 \pm \epsilon, t) \le v(x_0 \pm \epsilon, t), \quad 0 \le T_0. \tag{3.113}$$

Considering the expression (3.110) and replacing into (3.111):

$$\begin{aligned} &D|x-x_0|^{\alpha}\beta(T_0-t)^{\beta-1} - (m-1)D^2|x-x_0|^{2\alpha-2}(T_0-t)^{2\beta}\alpha(\alpha-1) \\ &- D^2\alpha^2|x-x_0|^{2(\alpha-1)}(T_0-t)^{2\beta} - \lambda D^{\theta+1}|x-x_0|^{\theta\alpha+\alpha-1}(T_0-t)^{2\beta}\alpha c \\ &- \mu|x-x_0|^{\sigma+\alpha\delta}(T_0-t)^{\beta\delta} \ge 0. \end{aligned} \tag{3.114}$$

To determine particular values for α and β consider:

$$\alpha = \max\{\alpha_1, \alpha_2\}, \tag{3.115}$$

$$\alpha_1 = 2\alpha_1 - 2 \rightarrow \alpha_1 = 2, \qquad \theta\alpha_2 + \alpha_2 - 1 = \sigma + \alpha_2\delta \rightarrow \alpha_2 = \frac{\sigma+1}{1+\theta+\delta}. \quad (3.116)$$

$$\beta = \frac{1}{1-\delta}. \quad (3.117)$$

Note that $\delta < 1$ then $\beta > 0$. For the sake of simplicity, consider the translation to $t = 0$, so that (3.114) reads:

$$\begin{aligned} D\epsilon^{\alpha}\beta T_0^{\beta-1} \geq & (m-1)D^2\epsilon^{2\alpha-2}T_0^{2\beta}\alpha(\alpha-1) + D^2\alpha^2\epsilon^{2(\alpha-1)}T_0^{2\beta} \\ & + \lambda D^{\theta+1}\epsilon^{\theta\alpha+\alpha-1}T_0^{2\beta}\alpha c + \mu\epsilon^{\sigma+\alpha\delta}T_0^{\beta\delta}. \end{aligned} \quad (3.118)$$

So that:

$$T_0^{\beta+1} \leq \frac{\epsilon^{\alpha}\beta}{(m-1)D\epsilon^{2\alpha-2}\alpha(\alpha-1) + D\alpha^2\epsilon^{2(\alpha-1)} + \lambda D^{\theta}\epsilon^{\theta\alpha+\alpha-1}\alpha c + \mu\epsilon^{\sigma+\alpha\delta}D^{\delta-1}}. \quad (3.119)$$

Note that we have required $T_0 < 1$, then it suffices to consider:

$$(m-1)D\epsilon^{2\alpha-2}\alpha(\alpha-1) + D\alpha^2\epsilon^{2(\alpha-1)} \geq \epsilon^{\alpha}\beta\frac{1}{2}, \ \rightarrow \ D_1 \geq \frac{\beta/2}{(m-1)\epsilon^{\alpha-2}\alpha(\alpha-1)+\alpha^2\epsilon^{\alpha-2}}. \quad (3.120)$$

In addition:

$$\lambda D^{\theta}\epsilon^{\theta\alpha+\alpha-1}\alpha c \geq \epsilon^{\alpha}\beta\frac{1}{4}, \ \rightarrow \ D_2 \geq \left(\frac{\epsilon^{\alpha}\beta}{4\lambda\epsilon^{\theta\alpha+\alpha-1}\alpha c}\right)^{\frac{1}{\theta}} \quad (3.121)$$

$$\mu\epsilon^{\sigma+\alpha\delta}D^{\delta-1} \geq \epsilon^{\alpha}\beta\frac{1}{4}, \ \rightarrow \ D_3 \geq \left(\frac{\epsilon^{\alpha}\beta}{4\mu\epsilon^{\sigma+\alpha\delta}}\right)^{\frac{1}{\delta-1}}. \quad (3.122)$$

We consider:

$$D = \max\{D_1, D_2, D_3\}. \quad (3.123)$$

□

In addition and for the case with $m \geq 2$, the expression (3.64) (for $v \rightarrow 0$) is an order one equation that can be solved along characteristics with the maximal solution:

$$W(x,t) = a\left(bt + r - \frac{1}{n}\right)_+, \ \ r = |x|, \ \ n \in \mathbb{N}. \quad (3.124)$$

Note that a and $b > 0$ are constants to assess. In particular, for $0 \leq \tau \leq T_0$ (where T_0 is given in (3.119)), $b\tau = \frac{1}{2n}$, where b shall be determined. Then:

$$W(x,t) \equiv 0 \quad \text{for} \quad r < \frac{1}{2n} \quad \text{and} \quad 0 \leq t \leq \tau. \quad (3.125)$$

Admit that any solution to (3.64) (with $v \rightarrow 0$) is bounded. Then:

$$v(x,t) \leq K_1 \ \text{ for } \ x \in \mathbb{R}, \ 0 \leq t \leq \tau \ \text{ and } \ K_1(p, \|v_0\|_{\infty}). \quad (3.126)$$

Our intention is to make $W(x,t)$ a maximal solution:

$$W(x,t) \geq v(x,t), \rightarrow a\left(bt + r - \frac{1}{n}\right)_+ \geq K_1. \quad (3.127)$$

Consider now any $r > \frac{1}{n}$. Assume $r = \frac{2}{n}$. so that for $t = 0$:

$$a\left(\frac{2}{n} - \frac{1}{n}\right)_+ \geq K_1, \quad a \geq nK_1. \tag{3.128}$$

We have built a supersolution $W(x,t) \geq v(x,t)$ in $r = \frac{2}{n}$ and $0 \leq t \leq \tau \leq T_0$. To determine b, the following condition holds:

$$W_t \geq |\nabla W|^2, \tag{3.129}$$

and considering that $W_t = ab$; $W_r = a$, then $b \geq a$.

According to expression (3.128) and the determined condition for a and b, $W(x,t)$ is a local supersolution:

$$W(x,t) \geq v(x,t), \quad 0 < |x| < \frac{2}{n}, \quad 0 \leq t \leq \tau \leq T_0. \tag{3.130}$$

The inequality (3.130) permits to conclude that any local supersolution satisfies the null criteria (or finite propagation speed) in the waiting time $0 \leq \tau \leq T_0$, then, any minimal solution $v(x,t)$ satisfies such null criteria in $0 \leq \tau \leq T_0$.

3.4. Applications to a fire extinguishing process

A flight test, with a fire extinguisher release, is required to determine each of the involved parameters in P. For this purpose, the nacelle has been split in four different compartments hosting the sensoring equipment (Figure 1).

Table 1 provides the agent volumetric concentration measured by each sensor. All measurements are given at a snapshot time of 3 seconds to ensure a well established discharge out of any initial turbulence. Note that the selection of the 3 seconds is particular to the discharge process modelled. In any other case, a different snapshot time may be selected depending on the required time for stabilized measurements.

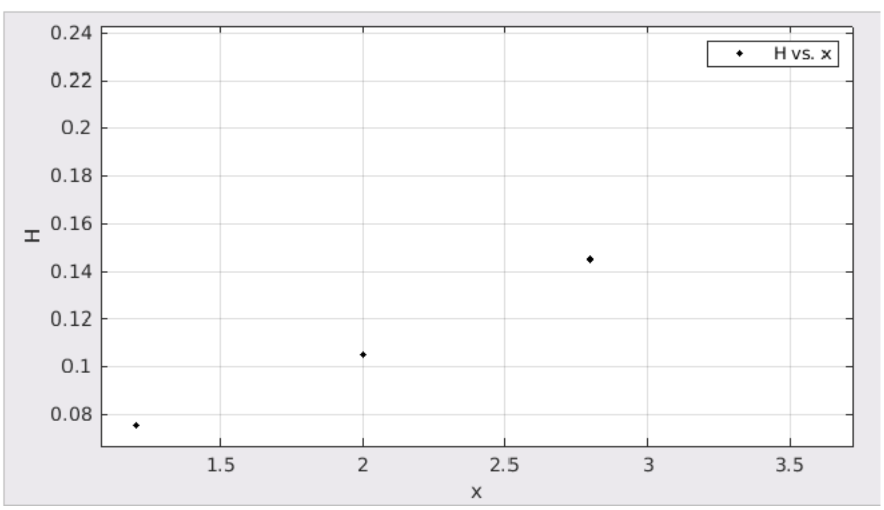

Figure 5. fire suppressant concentration (H) represented versus the axial variable x (meters)

Table 1. Suppressant agent concentration at t = 3s. This local in time behaviour is considered to determine the parameters involved in the model

Sensor section	longitudinal offset (m)	% Volumetric concentration at t = 3s
A	1,200	7,5
B	2,000	10,5
C	2,800	14,5
D	3,600	23,5

Once the fire alarm appears in the aircraft cockpit, the crew enables the fire automatic process and the bottles release the suppressant. At this moment, we consider that the reaction term is more significant than the non-linear diffusion. This assumption is sensible as the discharge time is qualitatively fast. The data in Table 1 is represented in Figure 5 and follows a potential law:

$$u = 0,1461|x|^{0,3277}, \tag{3.131}$$

which is used together with the expression (3.44). Note that we only consider the influence of the reaction (K_1):

$$\hat{u}_{\tau\to 0} = (1-p)^{\frac{1}{1-p}}|x|^{\frac{\sigma+1}{q-p}}(3)^{\frac{1}{1-p}}, \tag{3.132}$$

then, we have:

$$(1-p)^{\frac{1}{1-p}}(3)^{\frac{1}{1-p}} = 0,1461, \tag{3.133}$$

which provides a value for p:

$$p = 0,78. \tag{3.134}$$

Note that $p < 1$ as it was previously assumed. Additionally, we can obtain a value for q and σ considering:

$$\frac{\sigma+1}{q-p} = 0,3277. \tag{3.135}$$

As $\sigma > 0$, the previous expressions provides:

$$q > 3,832. \tag{3.136}$$

Then, we consider $q = 4$ so that:

$$\sigma = 0,055. \tag{3.137}$$

Considering that the reaction wins over the diffusion:

$$m < \left(\frac{\sigma+2}{\sigma+1}\right)(q-p) + p \quad \to \quad m < 7,0521 \tag{3.138}$$

then and as per expression (2.3)

$$m = \frac{4}{3}. \tag{3.139}$$

The existence of global solutions requires to check if $p \leq p^*$ as per Theorem 3.3. Note that p^* is given in the expression (3.47), then:

$$p \leq p^* \quad \to \quad 0,78 < 3,907. \tag{3.140}$$

Thus, there exists a global solution expressed by (3.44) that adopts the particular form:

$$\hat{u}(x,t) = 0,00102|x|^{0,3277}t^{4,545} \tag{3.141}$$

where

$$0 < u < 1, \tag{3.142}$$

and x (meters) and t (seconds) are the spatial and time variables.

The fire extinguishing starts with the fire bottles activation. As mentioned, the bottles discharge a pressurized suppressant that propagates along the domain [22]. We may be interested on finding the behaviour of the propagation front when $|t-t_0| << 1$, let say $|t-t_0| = 0,1$. For this purpose, we consider the expression (3.59) being $t_0 = 3$ seconds.

$$0 < A < t_0 \max_{x\in\mathbb{R}}\{v_0(x)\} + |x||\log t_0|^{-\frac{1}{2}} = |x||\log 3|^{-\frac{1}{2}} = 0,954|x|. \tag{3.143}$$

Then, the expression (3.59) reads:

$$|x| \leq 1,820|x|^2|\log(t-t_0)|^{\frac{3}{2}}, \tag{3.144}$$

such that

$$\frac{1}{|x|_{max}} \leq \frac{1}{|x|} \leq 1,820|\log(0,1)|^{\frac{3}{2}}. \tag{3.145}$$

The previous inequality is used to determine the support propagating envelope. For this purpose, we consider the equality:

$$|x|_{max} = 0,159m. \tag{3.146}$$

This value permits to consider the initial fire agent discharge spatial range. Initially (for a time of $|t-t_0| = 0,1$), the fire agent is limited to an scope of $0,159m$. For comparison purposes, note that Figure 6 provides a value of $4,012m$ for the maximum engine nacelle path.

Now, the objective is to determine the propagating front at discharge with $|t-t_0| >> 1$. For this purpose, we make use of (3.60). Previously note that:

$$1-p \geq m-q, \tag{3.147}$$

and according to (3.65) together with the particular values for m and p obtained:

$$\mu = 0,832\,; \quad \delta = 0,34. \tag{3.148}$$

Note that A is get upon the expression (3.143) assuming $|x| = 4,012m$ to account for the complete engine nacelle path (Figure 6):

$$0 < A < 0,954|x| = 0,954 \cdot 4,012 = 3,8274. \tag{3.149}$$

It suffices to consider $A = 3,827$. Then (3.60) reads:

$$|x| \leq 7,407\,t^{1,820}. \tag{3.150}$$

Accordingly, the fire suppressor required time to cover the nacelle ($|x| = 4,012m$ in Figure 6) is:

$$t = 0,714s. \tag{3.151}$$

Up to now, we have determined two different times with the corresponding spatial term along the support:

$$\begin{aligned} t &= 0,1s\,; \qquad |x| = 0,159m, \\ t &= 0,714s\,; \quad |x| = 4,012m. \end{aligned} \tag{3.152}$$

Initially, the propagating front moves slowly. Nonetheless, the positivity in the solution induced by the discharge makes the propagating front to accelerate and to complete the whole domain in a relative short time. Based on (3.152), we can assess a global value for the advection:

$$c = \frac{4,012}{0,714} = 5,619\,m/s. \tag{3.153}$$

In complex installations, such as the engine nacelles, there might be dead zones where the fire suppressor propagates by diffusion only (not by advection). A characterization on the behaviour of solutions is provided by Theorems 3.5 and 3.6. First of all, note that $q \geq m-1$ so that the expression (3.78) holds. The values of α and β are obtained according to (3.53):

$$\alpha = \frac{\sigma+1}{2\left((p-q)+(\sigma+1)\left(\frac{m-1}{2}\right)\right)} = -0,173, \quad \beta = q-p = 3,220. \tag{3.154}$$

Upon substitution of each of the parameters obtained:

$$\frac{\sigma(m-1)}{2(1-p)} = 0,8, \quad c_{supp} = \frac{(-\alpha+\beta d)^{\frac{m-1}{2(p-1)}}}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}} = 0,1845. \tag{3.155}$$

Then (3.78):

$$\xi_0 = 0,1845|x|^{0,8}. \tag{3.156}$$

Remind the expression (3.50) for ξ. Then, making the $|x|$ explicit and considering that $|x|_{dz}$ represents a characteristic dead zone length, we have:

$$|x|_{dz} = 0,00021\,t^{-16,2}. \tag{3.157}$$

In the hypothesis of $|x|_{dz} = 0,3m$, the propagating time required by the support to cover such dead zone is of $t = 0,6s$. Note that we consider the effect of the diffusion only, when propagating along the support which implies to have positivity ($u > 0$) whenever

$$|x| < 0,00021\,t^{-16,2}. \tag{3.158}$$

The required times for the fire suppressor to reach the whole nacelle are given in expressions (3.152). Note that the minimum required suppressor concentration to extinguish a fire is 6% in volume. Then, as per expression (3.141) with $u \geq 0,06$:

$$t = \left(\frac{u}{|x|^{0,3277}\,0,00102}\right)^{\frac{1}{4,545}} = 2,24s. \tag{3.159}$$

This value fits closely the data provided in Table 1 where the sensors measure more than 6% given a time higher than that assessed in (3.159). Furthermore,

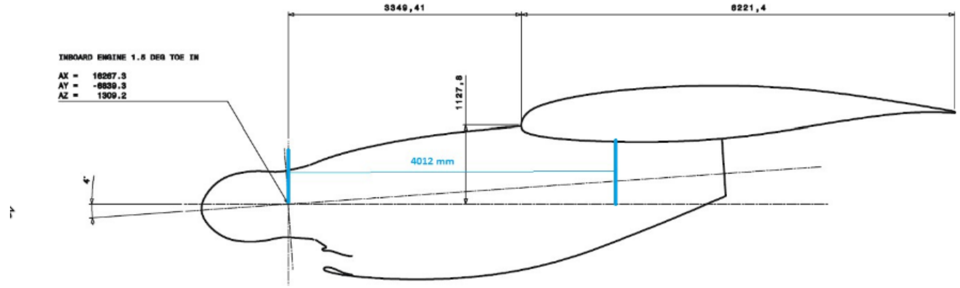

Figure 6. Nacelle shape geometry

returning to figure 4, it is possible to check that at $t = 2,24s$, all sensors (except sensor 8 located in a dead zone) are measuring above the level of 6% which permit to state the validity of the proposed assessment.

Finally, we determine the local supersolution to represent the finite propagation profile making use of Theorem 3.6. Assume that we are interested on knowing the solution profile in the mentioned dead zone. Then $\epsilon = 0,3\,m$ in Theorem 3.6. Once each of the parameters are assessed, it is possible to conclude on $T_0 \leq 0,510\,s$. The waiting time T_0 represents the maximal time for which the solution support is kept null. In other words, once the propagating fire suppressant reaches the interface of a dead zone, we shall wait for a maximum of $0,510s$ to see the agent populating such dead zone.

Now, we summarize the assessed times to have a full physical intuition of the fire suppressant process:

- Once the extinguisher bottle activates, the agent covers a distance of $|x| = 0,159m$ in $t = 0,1s$. This assessment permits to estimate the discharge energy.
- The suppressant propagating interface reaches the whole domain ($|x|=4,012m$) in a time of $0,714s$.
- Nonetheless, the propagating interface does not permit to extinguish a fire per se. For this, we need a minimum volumetric concentration of 6% for which a time of $2,24s$ is required.
- Finally, in the engine nacelle dead zones (assumed to be of a spatial mean value $0,3m$) the non linear diffusion is predominant. The required time for the diffusion to reach all areas in the dead zone is assessed to be $0,6s$. Nontheless and previous to enter the zone, the finite propagation feature induces a waiting time of maximum $0,510s$ to initiate the propagation. As a consequence, the required time for the diffusion to cover the whole dead zone is $0,6 + 0,510 = 1,11s$. Note that this time assessment is particularly relevant for the sensor 8 (refer to Figure 4) that is hosted in a zone with low advection. Note that if we consider the time required for the diffusion to cover the dead zone and extra-time shall be added to the time required to reach the 6% leading to a time of $3,35s$. Considering this values into Figure 4 since the beginning of discharge, it is possible to check that the sensor 8 measures above 6%.

4. Conclusions

The fire suppressant model, based on a non-linear diffusion, has been discussed with a mathematical approach to support a solid analytical approximation. A testing campaign on an aircraft has determined finite values for p, q, m and σ for which a global solution has been shown to exist. The assessments provided have permitted to answer global questions to understand the involved physics as a basis for engineering tasks. Mainly, the questions have consisted on assessing the required time to ensure the suppressant reaches the whole domain (including the dead zones) and to a sufficient level to extinguish a fire.

Conflict of Interest

The author states that there is no conflict of interest.

References

[1] P. Arturo and J. L. Vázquez, *The balance between strong reaction and slow diffusion*, Communications in Partial Differential Equations, 1990, 15, 159–183.

[2] P. Arturo and J. L. Vázquez, *Travelling Waves and Finite Propagation in a Reaction-Diffusion Equation*, Journal of Differential Equations, 1991, 93, 19–61.

[3] J. Bedrossian, N. Rodríguez and A. Bertozzi, *Local and global wellposedness for aggregation equations and patlak–keller–segel models with degenerate diffusion*, Nonlinearity, 2001, 24(6), 1683.

[4] J. M. Bennett, *Principles, Testing and in-field experience for the fire panel fuel tank protection device*, SAE, 2005.

[5] A. Bertozzi and D. Slepcev, *Existence and uniqueness of solutions to an aggregation equation with degenerate diffusion*, Communications on Pure and Applied Analysis, 2009, 9(6), 1617.

[6] M. Bertsch and D. Hilhorst, *A density dependent diffusion equation in population dynamics: stabilization to equilibrium*, SIAM Journal on Mathematical Analysis, 1986, 17(4), 863–883.

[7] M. Bhatti, A. Zeeshan, R. Ellahi, O. Anwar Bég and A. Kadir, *Effects of coagulation on the two-phase peristaltic pumping of magnetized prandtl biofluid through an endoscopic annular geometry containing a porous medium*, Chin. J. Phys. 2019, 58, 222–23. https://doi.org/10.1016/j.cjph.2019.02.004.

[8] D. Blake and J. Suo-Anttila, *Aircraft cargo compartment fire detection and smoke transport modeling*, Fire Safety Journal, 2008, 43(8), 576–582.

[9] A. De Pablo, *Doctoral Thesis. Estudio de una ecuación de reacción-difusión*, Universidad Autónoma de Madrid, 1989.

[10] J. Díaz, *Modeling of an aircraft fire extinguishing process with a porous medium equation.* SN Appl. Sci, 2020, 2, 2108. https://doi.org/10.1007/s42452-020-03891-9.

[11] L. Evans, *Partial Differential Equations*, Advanced Mathematical Society, United States of America, 2010.

[12] R. Ferreira, A. De Pablo, G. Reyes and A. Sánchez, *The interfaces of an inhomogeneous Porous Medium Equation with Convection*, Comm. in Partial Diff. Eq., 2006, 31, 497–514.

[13] E. R. Galea and N. C. Markatos, *A review of mathematical modelling of aircraft cabin fires*, Applied Mathematical Modelling, 1987, 11(3), 162–176.

[14] B. Gilding, *Improved theory for a nonlinear degenerate parabolic equation*, Ann. Scu. Norm. Sup. Pisa, 1989, 16, 165–224.

[15] P. Grasso, S. Mauro and A. Innocente, *Two-dimensional reaction-advection-diffusion model of the spread of fire in wildlands*, Book Chapter published in Advances in forest fire research, 2018, 334–342.

[16] R. Harish and K. Venkatasubbaiah, *Mathematical modeling and computation of fire induced turbulent flow in partial enclosures*, Applied Mathematical Modelling, 2013, 37(23), 9732–9746.

[17] S. A. Hosseini, N. Darabiha and D. Thévenin, *Lattice Boltzmann advection-diffusion model for conjugate heat transfer in heterogeneous media*, International Journal of Heat and Mass Transfer, 2019, 132, 906–919.

[18] S. Kamin and P. Rosenau, *Propagation of thermal waves in an inhomogeneous medium*, Comm. Pure and Applied Math, 1989, 34, 831–852.

[19] R. Kersner, G. Reyes and A. Tesei, *On a class of parabolic equations with variable density and absorption*, Adv. Diff. Equations, 2002, 7, 155–176.

[20] J. Kim, B. Baek and J. Lee, *Numerical analysis of flow characteristics of fire extinguishing agents in aircraft fire extinguishing systems*, J. Mech. Sci Technol., 2009, 23, 1877–1884.

[21] H. Li, *Hopf Bifurcation of Delayed Density-Dependent Predator-Prey Model*, Acta. Math. Sci, 2019, 39(2), 358–371.

[22] A. Maranguides, R. Sheinson, R. Darwin, D. Kay and D. Barylski, *Halon 1301 Retrofit Implementation Considerations*, Naval Researh Laboratory, Combustion Dynamics Section.

[23] S. Nazari, R. Ellahi, M. M. Sarafraz et al., *Numerical study on mixed convection of a non-Newtonian nanofluid with porous media in a two lid-driven square cavity*, J. Therm Anal Calorim, 2020, 140, 1121–1145. https://doi.org/10.1007/s10973-019-08841-1.

[24] C. Pao, *Nonlinear Parabolic and Elliptic Equations*, Springer Science+Bussiness Media, North Carolina, United States of America, 2012.

[25] T. A. Penteado, *Analysis of fire extinguishing agent concentration in commercial aircraft cargo compartment*, Master dissertation, Technological Institute of Aeronautics, 2004, 111.

[26] J. Santrock and S. E. Hodges, *Evaluation of automatic fire suppression system in full scale vehicle fire tests and static vehicle fire test*, SAE, 2004.

[27] A. Shahid, H. Huang, M. M. Bhatti, L. Zhang and R. Ellahi, *Numerical Investigation on the Swimming of Gyrotactic Microorganisms in Nanofluids through Porous Medium over a Stretched Surface*, Mathematics, 2020, 8, 380. https://doi.org/10.3390/math803038.

[28] J. L. Vázquez, *The Porous Medium Equation, mathematical theory*, Oxford Mathematical Monographs, Oxford, 2006.

[29] Y. Zhang, *Degenerate Diffusions with Advection*, Doctoral Thesis, University of California, 2019.

[30] Y. Zhou, *Exact Solutions and dynamics of the Raman Soliton model in Nanoscale Optical Waveguides, with metamaterials, having parabolic law non-linearity*, Journal of Applied Analysis & Computation, 2019, 9(1), 159–186. doi: 10.11948/2019.159.