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Research paper

Solving the chaos model-data paradox in the cryptocurrency market

Lukasz Pietrych[a], Julio E. Sandubete[b,c,*], Lorenzo Escot[b]

[a] Faculty of Economics, Warsaw University of Life Sciences Nowoursynowska 166, 02-787 Warszawa, Poland
[b] Faculty of Statistical Studies, Complutense University of Madrid, Av. Puerta de Hierro, 28040 Madrid, Spain
[c] Faculty of Economics, San Pablo CEU University, c/ Julián Romea 23, 28003 Madrid, Spain

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A B S T R A C T

In this paper we test for nonlinearity and chaos in some cryptocurrencies returns and volatility. Financial markets are characterized by the so-called *chaos model-data paradox*, that is, it is relatively easy to design theoretical dynamic financial models that behave chaotically, but it is hard to find robust evidence of this kind of chaotic behaviour in real dataset. In fact, this paradox has been taken as an evidence that support the Efficient Market Hypothesis (EMH). In this paper we apply new robust computational methods based on statistical procedures to reconstruct the underlying attractor and to estimate the Lyapunov exponents based on the Jacobian neural nets. We have tested nonlinearity and chaos in some digital cryptocurrencies (Bitcoin, Ethereum, Ripple and Litecoin). The results show strong evidence against EMH supporting the hypothesis that those time series come from an underlying unknown generating process that behave nonlinear and chaotically. This fact points out that a potential explication to the chaos model-data paradox lies in the methods traditionally used in the literature which are not robust and do not have the capability to find chaos in financial time-series data.

1. Introduction

In recent years, a new type of asset has been introduced to the financial markets, known as cryptocurrency (or digital currency). Bitcoin (BTC) is the most popular among cryptocurrencies, considering the level of market capitalization. Nowadays its capitalization is over 180 billion USD, and the value of daily volume is over 40 million USD[1]. Other popular currencies considering these features are Ethereum (ETH), Ripple (XRP) and Litecoin (LTC).

Despite the explicit development of digital currencies, the character and the core of cryptocurrency are not entirely clear yet. This state is conditioned by limited information on cryptocurrency exchanges, a decentralized money creation system, no savings accounts in cryptocurrency, and thus, no interest rates. Some authors e.g., [4] call digital currencies as “synthetic”, arguing that their creator is not a state i.e., central bank or government. In addition, digital currencies do not represent any hidden resource or other source of wealth. Of course, traditional currencies also do not have the gold standard that was abandoned in the 20th century, but in this case their value is like a barometer of the country’s economy. Cryptocurrencies do not meet this condition. What is more they are not accepted as a common medium of exchange, nor are they used as a

* Corresponding author
E-mail addresses: lukasz_pietrych@sggw.pl (L. Pietrych), jsandube@ucm.es (J.E. Sandubete), escot@ucm.es (L. Escot).

[1] This dataset was obtained from https://coinmarketcap.com/currencies.

unit of account. It is also difficult to assume that they can be used as a form of storing value (due to high volatility). Instead, they are an excellent asset for speculative purposes.

Kristoufek [58] states that the process of shaping cryptocurrencies prices cannot be explained by economic theories, i.e. the model of future cash flows, purchasing power parity or interest rate parity. Because there is no macroeconomic basis for digital currencies, due to their global character, the supply function is constant or follows a certain algorithm (e.g. bitcoin market), limited in advance. On the demand side, the market is not fueled by growing income in the national economy but only by the expected profits from the possession of the currency and its subsequent sale (speculative demand). There is no profit from the possession of the currency due to the lack of interest on the digital currency. The market is therefore dominated by short-term investors who follow trends. There is no fundamental basis for shaping "fair" prices, and investor moods is a key factor influencing the prices.

On the other hand, the economic perspective is still being developed. In the literature, often for empirical research, the Efficient Market Hypothesis (EMH) is indicated as the theoretical framework [30]. The existence of long-range memory is contrary to EMH, therefore, it has become necessary to generalize the theory of information efficiency of stock exchange markets in the version provided by Peters [68], who introduced Fractal Market Hypothesis (FMH), according to which the financial market is characterized by the presence of self-similar structures. This feature does not lead to a complete rejection of the theory of effective markets. However, FMH shows that the market stops being stable when its fractal structure disappears.

Most of the research is focused on the bitcoin market as the most capitalized cryptocurrency. Bariviera et al. [5] showed that, until 2014 the time series had a persistent behavior, whereas afterwards that year the behaviour of returns' time series has been consistent with white noise, which means that the bitcoin market is more information-efficient. An attempt to explain reasons for the change in the system dynamics was made by Urquhart [81]. In this study he found that the bitcoin market may be in the phase of moving towards an effective market, which can be justified by the fact that this market is in the phase of emerging market and therefore it should not be surprising to conclude inefficiency.

In the future, however, it can be expected that the bitcoin market will be more efficient because it will attract more investors. Brauneis and Mestel [10] argue that cryptocurrencies become less predictable/inefficient as liquidity increases. Nadarajah and Chu [64] continued work in this direction by making power transformation of Bitcoin returns, this time without rejecting the information efficiency hypothesis. Also, the previous research for eight most capitalized cryptocurrencies does not allow to accept the hypothesis concerning the efficiency of the cryptocurrency market, see [22].

The second group of research concerns the fractal market hypothesis, in which tools of the chaos theory are used. We can mention the papers written by Takaishi [77] and Lahmiri [59], who independently studied the multifractal properties of Bitcoins. Research has shown evidence that the Bitcoin time series exhibit multifractality, and their causes are temporal correlation and the fat-tailed distribution. In a similar study [33] observed an orderly correlation in the bitcoin market. In the study he also received a Hurst exponent greater than typical values obtained in the case of stock market or standard currency market research.

On the other hand, there are other alternatives in the literature on the existence of chaos in the time series of digital currency prices. [24] claims that although the blockchain system can be considered algorithmically complicated, it is unlikely to enter into a chaotic regime. According to Siddiqi [76], the presence of investors who only follow capital gains may cause chaos in the market. It is therefore advisable to limit their role. Al-Yahyaee et al. [2] compared the multifractality properties of bitcoin price time series to gold, shares and global currency markets. Their findings indicate that the bitcoin market is the most inefficient compared to others, and the global stock index is the least ineffective. The lack of clear regulations and oversight are potential causes for long-term memory process. Similar conclusions were found for eight forms of cryptocurrencies representing almost 70% of cryptocurrency market capitalization. The autocorrelations for returns decay quickly, while the autocorrelations for absolute returns decay slowly. The Hurst exponent for volatility is more volatile than that of the returns, while they all suggest the long-range dependence phenomena. Bouri et al. [8] indicate the possibility to predict Bitcoin price movements based on price information from the aggregate commodity index and gold prices.

Khuntia and Pattanayak [56] conducted research in the context of Adaptive Market Hypothesis (AMH). It was found that there are both effective and inefficient episodes on the market. Thus, the conclusion drawn from previous studies that the market is explicitly either effective or ineffective is false. The existence of behavioural bias and the creation of new occurrence can change efficiency. In a further study [57] confirmed that long memory exists in the time series studied and that it changes over time, what is consistent with AMH. What is more, the study found a force explaining the impact of trade volume on long-term memory when price movement is bearish. During the normal period, no effect of trade volume on long memory was detected. Jiang et al. [50] also investigated changes in long-term memory process on the bitcoin market and also detected long-term memory, a high degree of inefficiency, and the property that the market does not become more efficient over time.

In this paper we have extended the results proposed by Gunay and Kaşkaloğlu [43] applying the jacobian indirect methods instead of the direct methods for estimating the Lyapunov exponents from the major cryptocurrencies rates due to the following reasons: the traditional direct methods are not robust to the presence of (small) measurement noise because these estimators can produce a linear scaling region giving a wrong chaotic Lyapunov exponent value even for non-chaotic systems. Note that most real-world observed time-series data as the cryptocurrencies rates are usually noise-contaminated signals.

It also does not have a satisfactory performance in detecting existing nonlinearities on time-series data of moderate sample sizes and the asymptotic distribution of the Lyapunov exponent estimator does not exist, that is, there is not chance to make statistical inference about the direct Lyapunov exponent estimator and test if the estimated values are or not statistically significants. These jacobian indirect methods solve all drawbacks which belong to the direct methods as shown in this paper.

We have also provided a more robust alternative procedure to the method followed by Omane-Adjepong and Alagidede [67] for testing the hypothesis of chaos through the Lyapunov exponents in the Cryptocurrency Market. In our case, we have detected a chaotic behaviour inside the daily log returns from the major cryptocurrencies rates instead of the periodicity range of $2-4$ days or $4-8$ day band in which they have found. To do that, we have presented a consistent method and efficient algorithms based on the jacobian indirect methods. In this sense, we propose an analytical derivation procedure, rather than numerically of the jacobian needed for the estimation of the Lyapunov exponents. We have also extended the traditional heuristic approaches based on prescriptions to a statistical approach based on model selection procedures taking into account the Bayesian information criterion and leave one out cross-validation techniques as a robust approach for estimating the embedding parameters and the number of nodes in the neural net models needed for the estimation of the Lyapunov exponents.

At this point, we have provided a discussion about the optimal sample size used for estimating the partial derivatives of the jacobian and the optimal block length considered which is the number of evaluation points used for estimating the Lyapunov exponents. We have discussed the use of full sample estimation and three different blocking methods for estimating the Lyapunov exponents. One of them is a new proposal based on the bootstrap method. Then a feasible test statistics were introduced for testing the hypothesis of chaos based on the theoretical asymptotic properties. We have also provided new algorithms implementing this formal test to make statistical inference.

The subsequent parts of the paper are organised as follows. Section 2 provides a discussion about the chaos model-data paradox in financial markets. Section 3 reports an empirical analysis for testing the time dependence and nonlinearity inside the four major cryptocurrencies rates. Section 4 presents a robust analysis of chaos from the cryptocurrency time series considered. Finally, Section 5 provides some concluding remarks.

2. The chaos model-data paradox in financial markets

It is often said that chaos is an ubiquitous phenomenon which is observed everywhere, in any branch of science e.g., [18,23,44]. In particular, in economics, it is more than three decades ago that ideas about chaos started to appear in the literature showing that it is possible to design economic models in a chaotic behaviour regime from a theoretical point of view, see e.g., [17,31,40,48,54]. Nevertheless, there is no clear evidence that economic time series are chaotic in their behaviour. So far, researchers have found strong evidence of nonlinearity, but little proof of chaos see e.g., [6,29,45,60,66,74].

This phenomenon was proposed by Brock et al. [12] as the *chaos model-data paradox*. This paradox shows that although it is relatively simple to formulate chaotic models with high economic plausibility (chaos is profuse in theoretical economic models), there is no clear evidence that economic time series behave chaotically (chaos is elusive in economic empirical data). Particularly in finance, beyond the hypothesis of efficient markets, the chaotic dynamics of prices can be theoretically modelled by resorting to the behavior of chartists and fundamentalists operating in financial markets, see e.g., [19,20,42]. Chartists expect the price to go up (or down) if it has gone up (or down) in the past, and fundamentalists will buy (or sell) when the market price is below (or above) its fundamental value. We can illustrate this behaviour following a simple example, see [31]:

$$P_{t+1} = P_t^{\alpha} e^{\beta (P_F - P_t)} \tag{1}$$

$\alpha \geq 1$ and $\beta > 0$ reflect the behaviour of chartists and fundamentalists respectively. When in the market there are no chartist strategies, $\alpha = 1$, the model will describe a dynamic transition towards the stationary state P_F, the fundamental price. When the two types of agents interact in the market, $\alpha > 1$, other types of dynamic solutions can appear, including chaotic behaviours, see Fig. 1. This is just a simple example to illustrate how easy it is to build theoretical models in finance that result in chaotic behaviour. The main scientific challenge therefore lies in the analysis of the financial empirical data [14]. It is quite difficult to detect chaos from the financial time series. There is substantial evidence for nonlinearity but relatively weak evidence for chaos. The appearance of cryptocurrencies provides a new type of financial asset, a new opportunity to test for chaotic behaviour, or at least, for nonlinear dynamics.

3. Nonlinearity analysis inside some cryptocurrency time series

3.1. Data description

In this paper we have considered the four major cryptocurrency rates with highest capitalization from the Cryptocurrency Market. We have chosen the data of daily frequency and the following periods: Bitcoin (BTC) and Litecoin (LITE) from April 29, 2013 to January 31, 2020 (2.469 observations). Ethereum (ETH) from August 7, 2015 until January 31, 2020 (1.639 observations). Ripple (XRP) from August 4, 2013 to January 31, 2020 (2.372 observations). These periods were selected arbitrarily without any economic criteria. We selected all the time series until the end of January 2020 because later we experienced

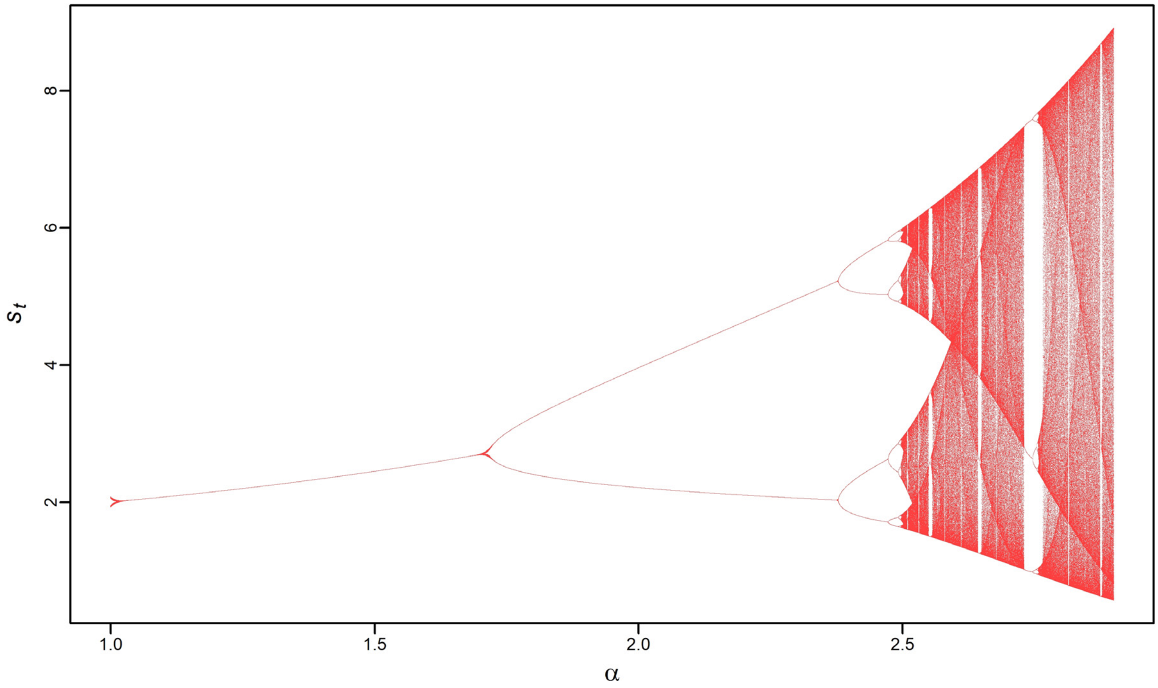

Fig. 1. Graphical representation about the bifurcation diagram from the solution by the map $P_{t+1} = P_t^{\alpha} e^{\beta(P_F - P_t)}$ with $\beta = 1$, $P_F = 2$, and $1 \le \alpha \le 3$.

an exogenous shock connected to the COVID-19 pandemic. This dataset was obtained from www.coinmarketcap.com which is a widely recognized price aggregator commonly used by the scientific community in this area. This financial market is characterised by being open typically from 8 a.m. to 4 p.m. local time.

Cryptocurrency's price relative to the US dollar, especially over the past four years, has a specific growth rate that cannot be seen with other financial assets. It can be seen that cryptocurrencies were subjected to a few downturns, but nevertheless proved to be resistant. At the beginning of 2014, the largest bitcoin exchange - Mt.Gox collapsed. Later, there was a period of fluctuation and stabilising of prices, until to the 2017, when the cryptocurrency's value changed exponentially, see Fig. 2. At the end of 2017, exchange rates increased during the month from around 100% for ETH to as much as 1000% for XRP. It was the largest speculative bubble in cryptocurrency history (a similar scale of revaluations of cryptocurrency's values took place at the turn of 2013 and 2014).

The high scale of cryptocurrencies exchange rate volatility in this period is probably due to the fact that it is primarily a speculative commodity. This can be confirmed by observing the frequency of searches for "bitcoin" on Google Trends. Byström and Krygier [15] found the significant link between Google search volumes and market volatility points at retail investors, rather than large institutions, being the most important drivers of cryptocurrencies volatility. From September 2017, the Chinese authorities began to put pressure on the managers of cryptocurrency platforms. The registration of new recipients was ordered to cease and the deadline for termination of activity was set. These activities determined the gradual closing of platforms serving crypto trade. Due to China's dominant position in cryptocurrency's mining, this policy negatively affected investors' moods and a fall in prices in 2018.

Before presenting the empirical results on chaos, we present descriptive statistics, test for normality and conclusions regarding the volatility in the BTC, ETH, LITE, and XRP price time series. We need to differentiate the log price series p_t in order to get the returns as $r_t = \log(p_t) - \log(p_{t-1})$. Fig. 3 describes the time evolution of the Bitcoin, Ethereum, Litecoin, and Ripple returns over the last 5 years. These graphs shows that return rates exhibit volatility clustering, which means that both large and small rate changes tend to occur in series over time. This implies instability of variance over time. This behaviour indicates that there may be a nonlinear dynamic structure in these time series data. Prices do not show a global trend in the period considered. Despite many changes, the price has always shown a tendency towards its mean.

As one can see in Table 1 the daily return time series have a standard deviation between 0.04 and 0.062, and very small average, close to 0, suggesting the high volatility for the periods considered. Despite the removal of extreme outliers, the values of the remaining statistics showed a deviation from the skew and kurtosis from the normal distribution, which means the occurrence of the leptokurtosis effect and fat tails of the distribution rates of returns. The probability of reaching an extremely high absolute value of a change in the share price is greater than for a normal distribution. The Jarque-Bera statistics also rejected the hypothesis of normal distribution in all the cryptocurrencies considered.

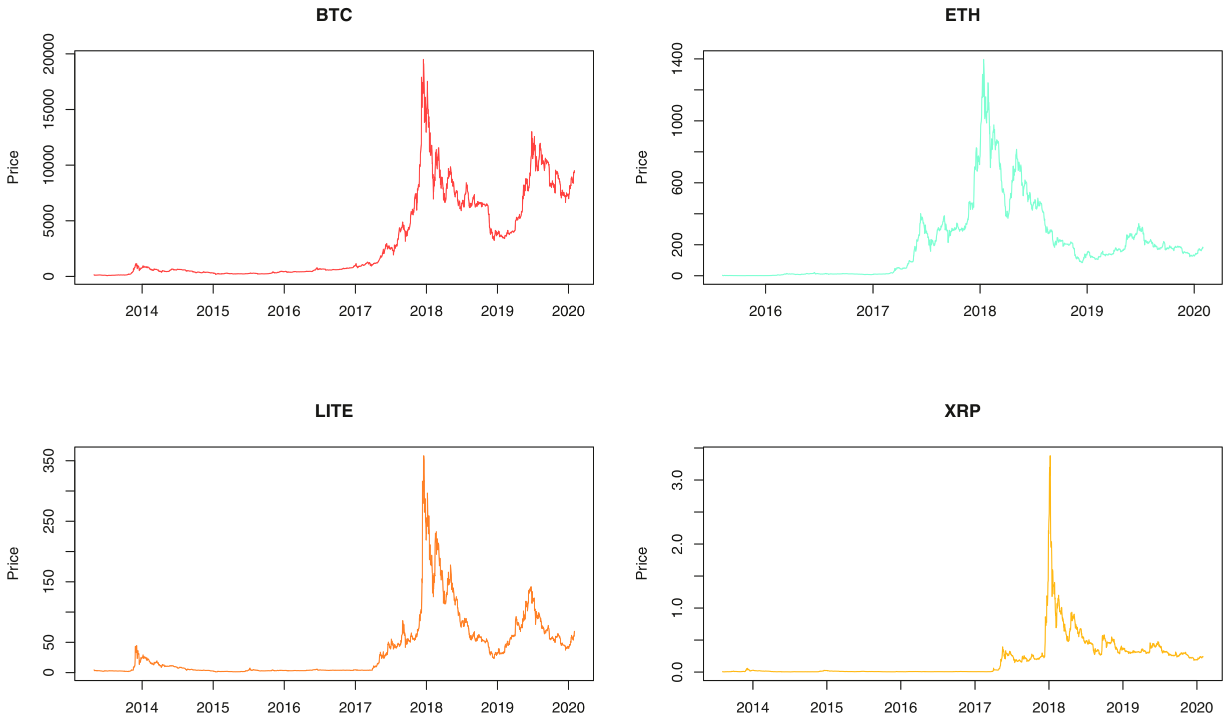

Fig. 2. Graphical representation of the time evolution of the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) prices over the last 5 years. The values shown are daily closing prices.

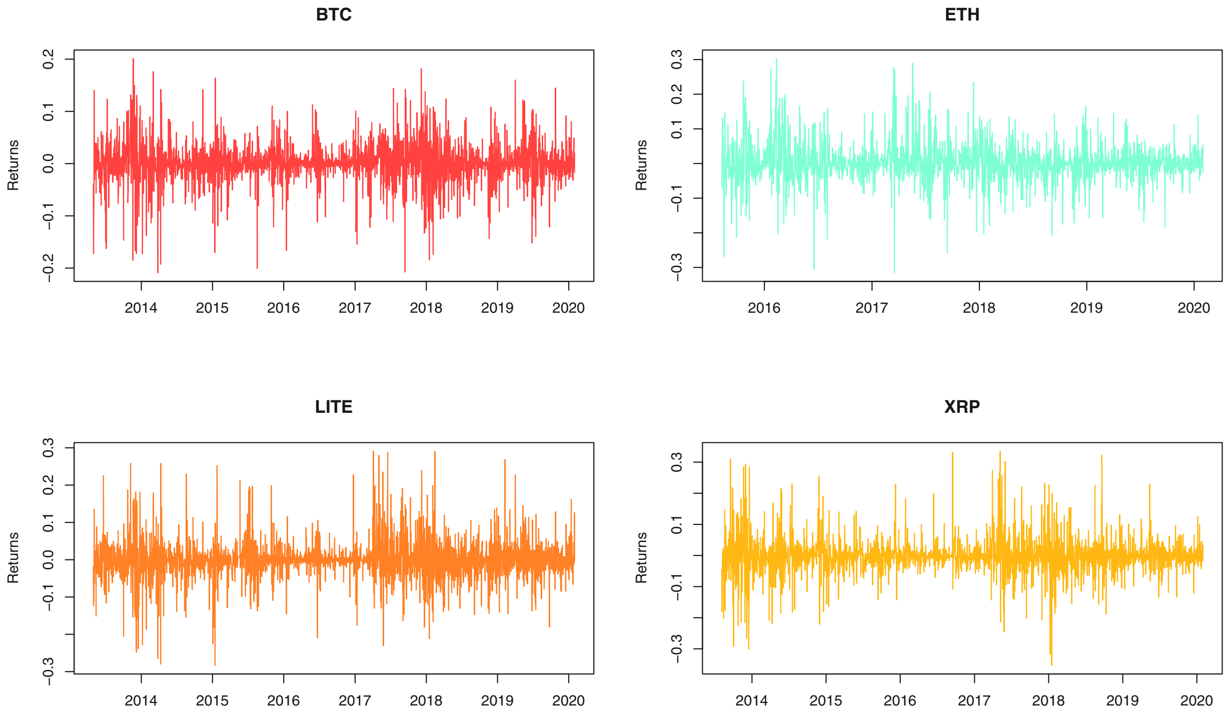

Fig. 3. Graphical representation of the time evolution of the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) returns over the last 5 years. The values shown are daily returns.

Table 1
Summary of the descriptive statistical analysis of the daily returns time series from the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) rates over the last 5 years. The number of observations provided are after removing outliers from time series.

	Bitcoin (BTC)	Ethereum (ETH)	Litecoin (LITE)	ipple (XRP)
Observations	2456	1630	2374	2352
Min	−0.209	−0.315	−0.283	−0.353
Max	0.201	0.303	0.291	0.335
Mean	0.002	0.003	0.000	0.000
Standard Dev.	0.040	0.062	0.056	0.060
Skewness	−0.383	0.199	0.364	0.527
Kurtosis	4.406	3.932	5.139	6.463
Jarque-Bera (*p*-value)	0.000	0.000	0.000	0.000

Table 2
Summary of the stationarity, time-dependence and nonlinearity analysis of the daily returns time series from the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) rates over the last 5 years. We have marked with an asterisk $*$ those Hurst exponents that are statistically significant with a p-value < 0.001.

Testing stationarity in mean	Bitcoin (BTC)	Ethereum (ETH)	Litecoin (LITE)	Ripple (XRP)
ADF (H_0: *the series possesses an unit root then it is not stationary)*				
5% Critical Value:	−1.940	−1.940	−1.940	−1.940
Test-stat:	−2.058	−1.411	−5.186	−3.153
KPSS (H_0: *the series is stationary)*				
5% Critical Value:	0.463	0.463	0.463	0.463
Test-stat:	0.076	0.343	0.262	0.055
Testing time-dependence in mean				
Ljung-Box Q (20 lags) (H_0: *no serial correlation)*				
p-value:	0.010	0.001	0.000	0.000
Hurst exponent:				
H_{DFA}	0.584*	0.625*	0.550*	0.524
H_{DFA} (shuffled data)	0.506	0.505	0.518	0.502
Testing nonlinear time-dependence in mean				
Surrogate Test (H_0: *the series is a gaussian linear process)*				
p-value:	< 0.05	< 0.05	< 0.05	> 0.05
Teraesvirta Test (H_0: *linearity in 'variance')*				
p-value:	0.000	0.000	0.000	0.000
White Test (H_0: *linearity in 'variance')*				
p-value:	0.000	0.000	0.002	0.000
BDS Test (ARMA fit) (H_0: *standarized residuals are i.i.d. random variables)*				
p-value:	0.000	0.000	0.000	0.000
BDS Test (shuffled data) (H_0: *standarized residuals are i.i.d. random variables)*				
p-value:	> 0.1	> 0.1	> 0.1	> 0.1

3.2. Testing stationarity, time-dependence and nonlinearity

Stationarity was verified based on two widely used unit root tests by Dickey-Fuller test (ADF) and Kwiatkowski, Phillips, Schmidt and Shin test (KPSS). Table 2 shows the results of the unit root tests. The ADF and KPSS test statistics and the corresponding critical values suggest the conclusion that the log return time series is stationary. Only for ETH, ADF test show the series are nonstationary.

Seeking linear dependence based on Ljung-Box statistics (20 lags), the null hypothesis (no serial correlation) is rejected. Similarly, McLeod-Li statistics reject the null hypothesis (no serial correlation in square series) and confirm heteroscedasticity in return series, suggesting that there is a certain kind of nonlinear relationship in square series. The above conclusions have also been confirmed by Engle's ARCH test, see Table 3.

Diagnosing the long memory process is an important step, so we performed Hurst exponent using detrended fluctuation analysis (DFA). However the estimation of the Hurst exponent is not substantial. We also need a measure for the significance of the results. We found evidence for long-range dependence in returns for BTC, ETH, and LITE - significant at the two-sided 99% level. We compute the Hurst exponent not only for the original data, but also for the shuffled time series to remove long-range correlations in the data and achieve more stronger evidence for nonlinearities in time series. Shuffle time series is generated by randomly shuffling the time-order of the original time series, so any temporal dependence that may have been in the data are broken. If Hurst exponent are different between the original and surrogate data suggests us that exist time dependence. We get clear results about time-dependence in mean because the Hurst exponent is not statistically significant when we consider shuffled time series, see Table 2. Evidence for long-range dependence in the variance is more significant, but it should be noted that in this case, data after shuffle didn't lose intrinsic long-range correlations of the original data, because the main source of correlations is the broad probability density function, see Table 3.

Table 3
Summary of the time-dependence and nonlinearity analysis of the daily volatility time series from the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) rates over the last 5 years. We have marked with an asterisk $*$ those Hurst exponents that are statistically significant with a p-value < 0.001.

	Bitcoin (BTC)	Ethereum (ETH)	Litecoin (LITE)	Ripple (XRP)
McLeod-Li (20 lags) (H_0: *no serial correlation in square series)*				
p-value:	0.000	0.000	0.000	0.000
Testing time-dependence in variance				
Engle ARCH (H_0: *the residuals are homoscedastic)*				
p-value:	0.000	0.000	0.000	0.000
Hurst exponent:				
H_{DFA}	0.880*	0.839*	0.864*	0.845*
H_{DFA} (shuffled data)	0.845*	0.836*	0.820*	0.810*
Testing nonlinear time-dependence in variance				
Surrogate Test (H_0: *the series is a gaussian linear process)*				
p-value:	< 0.05	< 0.05	< 0.05	< 0.05
Teraesvirta Test (H_0: *linearity in 'variance')*				
p-value:	0.331	0.000	0.057	0.000
White Test (H_0: *linearity in 'variance')*				
p-value:	0.225	0.009	0.067	0.000
BDS Test (GARCH fit) (H_0: *standarized residuals are i.i.d. random variables)*				
p-value:	0.000	0.000	0.000	0.000
BDS Test (shuffled data) (H_0: *standarized residuals are i.i.d. random variables)*				
p-value:	> 0.1	> 0.1	> 0.1	> 0.1

Before we use some methods of detecting nonlinear dynamics, which are part of the chaos theory, we must first answer the basic question whether the use of such advanced tools has its justification in the data. First, we use the surrogate test according to Schreiber and Schmitz [73] and Theiler et al. [80]. The surrogate data test for nonlinearity is one of the most popular, especially with regard to the null hypothesis that the examined time series is generated by a Gaussian (linear) process. Properly designed surrogate data for this null hypothesis must be completely random but it should possess the same autocorrelation and amplitude distribution as the original data. The surrogate data in our study is generated by using a phase randomization procedure. Tables 2 and 3 show that the null hypothesis which states that the data was generated by a Gaussian linear stochastic process with constant coefficients, is rejected for returns and variance, and original data's statistic is significantly smaller than surrogates' statistics. Similarly, we rejected the null hypothesis about linearity in mean based on the results obtained from the Teraesvirta test and White test.

The BDS test proposed by Broock et al. [11] is a test for checking the time independence in time series based on the correlation integral. It can be used for testing nonlinearity if firstly we removed the linear structure from data by prefiltering (fitting a general ARMA model) and then using the BDS test to check for remaining any nonlinear dependence in the residuals [32]. The results obtained are unambiguous, see Table 2. The rejection of the null hypothesis is strong, for $m = 2, \cdots, 10$ and all epsilon values for close points.

To enhance the evidence of non-linearity the BDS test is applied to shuffled residuals. As shuffling eliminates the alleged (non-linear) dependence of the data, the results provided by the BDS test should be difference between the shuffled and original residuals time series. In fact, Table 2 clearly show that the time dependence in the original data clearly disappear in the shuffled time series residuals at 99.9% confidence level.

Finally, we have also used the BDS test for checking any kind of nonlinear structure in the variance from the squared residuals provided by fitting a general GARCH model. As you can see in Table 3 the BDS test rejects the null hypothesis that the squared standardised residuals are i.i.d. random variables. That fact confirms that GARCH process does not capture all the nonlinearity in the square returns series. We have also found that the shuffled square standardized residuals show robust evidence of nonlinear dependence in variance.

We have already shown some strong results about the time-dependence in mean and variance of the four major cryptocurrencies and during the period considered. We have also provided some evidences to support the presence of nonlinearities in those cryptocurrencies. These are necessary conditions Now we would like to know if this nonlinear time-dependence shows a chaotic behaviour or not. In this sense, we are going to provide a new computational procedure which enables to detect robustly chaotic signals inside the cryptocurrencies returns and variance time series considered, if there were any.

4. How to detect a chaotic signal robustly? Direct vs Jacobian indirect methods

We have shown at the beginning that it is relatively easy to find economic models under chaotic behaviour from a theoretical perspective. However, there is no strong support for the idea that economic time series behave chaotically. In particular, in the cryptocurrency market researchers have found significant evidence of nonlinearity, but little proof of chaos so far. In this paper we provide a plausible explanation for this model-data paradox.

Our main motivation is that chaos is not usually find in cryptocurrency time series data because most recently published studies in this field tend to apply incorrect and not robust estimation methods. This could make it difficult to detect chaos in these time series.

It has been developed several techniques for estimating the complexity of time-series data, for a review see e.g., [9,28,79]. In this paper we have focused on methods derived from chaos theory that estimate the complexity of a data set by exploring the structure of the attractor. In particular, we have been interested in the so-called *Lyapunov exponent* (λ_z) as an attractor invariant measure. Quantifying chaos through this measure is a key point for understanding a chaotic behaviour. Therefore, we will be interested in testing the chaos hypothesis defined as follows:

$$\begin{aligned} &H_0 : \hat{\lambda}_z > 0 \\ &H_1 : \hat{\lambda}_z \leqslant 0 \end{aligned} \tag{2}$$

for some $z = 1, 2, 3, \ldots$ on a z-dimensional system. Reject the null hypothesis $H_0 : \hat{\lambda}_z > 0$ means absence of chaotic behaviour, see e.g., [34]. This quantitative measure can be described as following. Let $Y_T = F(Y_{T-1})$ be a difference equation where $F : \mathbb{R}^z \to \mathbb{R}^z$ for $T = 1, 2, 3, \ldots, N$. For a z-dimensional system there will be z Lyapunov exponents which are given by

$$\begin{aligned} \lambda_z(Y_0) &= \lim_{T\to\infty} \frac{1}{T} \{\log(|DF(Y_T)| \cdot \ldots \cdot |DF(Y_0)|)\} \\ &= \lim_{T\to\infty} \frac{1}{T} \left\{\log\left(\left|DF^T(Y_0)\right|\right)\right\} \end{aligned} \tag{3}$$

This ratio gives the mean rate of convergence or divergence of an orbit $F^T(Y_0)$ starting at point Y_0 where $DF^T(Y_0)$ is the jacobian evaluated along the trajectory $\{Y_0, Y_1, \ldots, Y_N\}$. Methods related to test the hypothesis of chaos try to estimate the Lyapunov exponents as a way of characterising a chaotic system. When the dynamic system is known, we can directly calculate the Lyapunov exponent of the system applying the own analytical equation that defines them (Eq. (3)). If we get a Lyapunov exponent greater than zero we can say that we have detected chaos because it shows sensitivity to initial conditions considering that the system is dissipative.

However, in most cryptocurrency time series we do not have the advantage of knowing the functional form F that generates the dynamics associated with it. Instead, there is a *observer function* $f : \mathbb{R}^{z+1} \to \mathbb{R}$ which transforms the unobserved state variable of the system into observed data. This function f generates observations as $y_t = f(Y_t)$ where the observable variable $y_t \in \mathbb{R}$ and $t \in \mathbb{Z}^+$ (discrete-time) for $t = 1, 2, 3, \ldots, n$. Then, we assume that all the available information is the time series $\{y_t\}_{t=1}^n$.

4.1. State space reconstruction from cryptocurrency time series

As the true data-generating process is unknown it will not be possible to consider the true orbit of the dynamic system in the original state space. Instead of that we have to get an approximation (reconstruction) of it that result equivalent in a topological sense. We mean equivalent in the dynamic and geometric properties. So this reconstruction procedure allows us to get all the significant information about the unknown data-generating process as the topology of the state space, the number and types of periodic orbits or certain invariant ergodic measures e.g., Lyapunov exponents.

The Lyapunov exponents defined above should have about the same value in both the true and reconstructed state space. This is an important result proposed by Takens [78]. This fact allows us to test the hypothesis of chaos (Eq. (2)) in the unknown original dynamic system. The embedding process provides a framework to reconstruct any unknown dynamic system which gave rise to a given observed scalar time series simply by reconstructing a new state space out of successive values of the time series.

We have considered the delayed-coordinates method proposed by Ruelle and Takens [70] in order to obtain the delayed-coordinate embedding vectors as following. Let $\{y_t\}_{t=1}^n$ be a time series. We set up a sequence of retarded vectors by associating for each time period a vector in a reconstructed state space $\mathbb{R}^m$, the coordinates of which satisfy the following equation:

$$y_t^m = \left(y_t, y_{t-\tau}, y_{t-2\tau}, \ldots, y_{t-(m-2)\tau}, y_{t-(m-1)\tau}\right) \tag{4}$$

where m is the *embedding dimension* and τ is the *reconstruction time-delay* (or lag). Note that despite the fact we have measured just a single observable we can construct a vector space whose axes represent all the relevant variables given by

$$\begin{aligned} y_1^m &= \left(y_1, y_{1-\tau}, \ldots, y_{1-(m-2)\tau}, y_{1-(m-1)\tau}\right) \\ y_2^m &= \left(y_2, y_{2-\tau}, \ldots, y_{2-(m-2)\tau}, y_{2-(m-1)\tau}\right) \\ &\vdots \\ y_n^m &= \left(y_n, y_{n-\tau}, \ldots, y_{n-(m-2)\tau}, y_{n-(m-1)\tau}\right) \end{aligned}$$

Note that a key issue in creating a proper state space reconstruction is to set the right criteria to make a coherent choice about the embedding parameters (τ and m). The consequences of making a wrong choice would be that the dynamic and geometric properties would not be the same between the true and the reconstructed state space. That is, the Lyapunov exponents in that case would not have the same value in both the true state space and the reconstructed one.

Researchers in this area usually estimate them using heuristic approaches and prescriptions that are mainly based on physical or geometrical arguments. As far as the estimation of the time delay τ is concerned, although other criteria exist, see e.g. [1,53], $\tau = 1$ is usually used following [78]. Regarding the embedding dimension m most of the published works consider the false nearest neighbour criterion provided by Kennel et al. [55]. the scientific community also often considers the correlation dimension as a proxy of the embedding dimension using the algorithm proposed by Grassberger and Procaccia [41].

The main disadvantages of these heuristic approaches are the following: (i) they are not inherently statistical; (ii) they give rise to estimators whose properties are unknown; (iii) they do not consider the output of any tuning of the model. In this sense, we have extended these traditional techniques to more robust procedures when it comes to estimate consistently the embedding parameters. Particularly, we have focused on a statistical approach which solves the main drawbacks of those heuristic approaches, see [16].

The aim is to choose *together* (not independently) the time-delay τ and the embedding dimension m which gives the optimal adjustment in the estimation of the Lyapunov exponents considering some information criteria as the Akaike Information Criteria (AIC) or the Bayesian Information Criterion (BIC). We have also considered the selection method based on cross-validation techniques which split the dataset into training, validation and test set.

Particularly, we have focused on the leave one out cross-validation technique which it is the most recommended cross-validation alternative in the reconstruction procedure. Keep in mind that the embedding parameters belong to the parameter set to be estimated when selecting the best model by several estimation methods of the Lyapunov exponent as we will show below.

4.2. Estimating Lyapunov exponents from cryptocurrency time series

Moving on once we have reconstructed the attractor we want to know if the unknown data-generating process shows a simple dynamic, behaved chaotically or came from a purely stochastic process. Methods and techniques related to testing the hypothesis of chaos (Eq. (2)) try to quantify the initial-value sensitive property estimating the Lyapunov exponents. There are two main methods in the literature that provide the estimated Lyapunov exponent from time series.

The first one, the so-called *direct* approach which directly measures the growth rate of the divergence between two nearby trajectories. The second one, the so-called *indirect* approach which try to estimate the Jacobian of the underlying generating system and then those partial derivatives are used to compute the Lyapunov exponent. We will discuss both methods although we have focused in greater detail into the Jacobian indirect methods because provide us consistent estimators and robustness as we will show later. Let us begin by focusing on the traditional direct methods.

The direct method was first proposed by Rosenstein et al. [69], Wolf et al. [83], and Kantz [51]. Hegger et al. [46], Kantz and Schreiber [52] describe the algorithm of this approach. These traditional direct methods have certain relevant disadvantages: (i) it allows just the estimation of the largest Lyapunov exponent; (ii) it is not robust to the presence of a small measurement noise because these estimators assign to chaos any divergence, even if purely random, caused by the measurement error itself; (iii) does not perform satisfactorily in detecting non-linearities in time series data of moderate sample sizes; (iv) there are not available any kind of theoretical results for its consistency and asymptotic distributions so far. Then there is not chance to make statistical inference about chaos. This is a strong disadvantage from a statistical perspective.

Other methods that do not based on estimating an ergodic measure such as the Lyapunov exponent to detect chaotic behaviour like the 0-1 test proposed by Gottwald and Melbourne [35–39] work very well when the system is purely deterministic. However, they tend to fail and have the same problems as the direct methods when the underlying generating system of the time series is unknown because it is not able to distinguish when an erratic time series is purely random or comes, at least partially, from a chaotic deterministic behaviour see e.g., [49] or [62].

For these reasons we have focused on the Jacobian indirect methods which solve all drawbacks that belong to the traditional direct methods. The idea behind this approach can be summarising briefly as follows. Firstly you have to estimate the Jacobian of the underlying generating system from the delayed-coordinate embedding vectors. Then you have to compute the Lyapunov exponent from those partial derivatives.

This methodology based on nonparametric regression methods was provided by Eckmann and Ruelle [25]. Other contributions focus on two different approaches. First, those using local linear regression techniques, see e.g., [13,26,72] and local polynomial regression methods, see [61]. Secondly, other approaches use non-linear neural network models, see e.g., [7,21,63,65,75,82].

Remember that our aim is to see how we can estimate robustly the Lyapunov exponent and how we can make inference in order to know if they are statistically greater than 0 or not. To do this based on the Jacobian indirect methods we need to evaluate the derivative along the whole trajectory because those derivatives will tell us how two closed initial trajectories diverge. Then we have to see how we estimate the Jacobians and evaluate them along the trajectory. We have considered the Jacobian indirect methods using the global neural net approach. Let us remember that we have a data-generating process which is unknown a priori, so we deal with an observer function which contain a measurement noise term. In that context, we have to get a pseudo dynamic system $G : \mathbb{R}^m \to \mathbb{R}^m$ such that $\mathbf{y}_t^m = G\left(\mathbf{y}_{t-\tau}^m\right)$ following the ideas proposed by Dechert and Gencay [21] where $\mathbf{y}_t^m$ are the delayed-coordinate embedding vectors (Eq. (4)). Following the ideas proposed by Takens [78] the Lyapunov exponents of F (from the original unknown dynamic system) and G should be the same, so we can focus on estimating the exponents from G. The dynamic system G may be expressed as a matrix which depends on a function υ and that function depends on the delays,

$$\begin{pmatrix} y_t \\ y_{t-\tau} \\ \vdots \\ y_{t-(m-2)\tau} \\ y_{t-(m-1)\tau} \end{pmatrix} = G \begin{pmatrix} y_{t-\tau} \\ y_{t-2\tau} \\ \vdots \\ y_{t-(m-1)\tau} \\ y_{t-m\tau} \end{pmatrix}$$

$$\begin{pmatrix} y_t \\ y_{t-\tau} \\ \vdots \\ y_{t-(m-2)\tau} \\ y_{t-(m-1)\tau} \end{pmatrix} = \begin{pmatrix} \upsilon\left(y_{t-\tau}, y_{t-2\tau}, \dots, y_{t-(m-2)\tau}, y_{t-(m-1)\tau}, y_{t-m\tau}\right) \\ y_{t-\tau} \\ \vdots \\ y_{t-(m-2)\tau} \\ y_{t-(m-1)\tau} \end{pmatrix} \tag{5}$$

The Jacobian based on the reconstructed dynamic system G will be got as follows,

$$\partial G = \begin{pmatrix} \frac{\partial \upsilon}{\partial y_{t-\tau}} & \frac{\partial \upsilon}{\partial y_{t-2\tau}} & \frac{\partial \upsilon}{\partial y_{t-3\tau}} & \cdots & \frac{\partial \upsilon}{\partial y_{t-(m-1)\tau}} & \frac{\partial \upsilon}{\partial y_{t-m\tau}} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{6}$$

Then the key to this approach is that the estimation of the Lyapunov exponent reduces to the estimation of the unknown non-linear function $\upsilon : \mathbb{R}^m \to \mathbb{R}$. That is, we have to estimate that function υ which collect the dependence between y_t and its past. As we said before the different approaches that make up the indirect method differ in the algorithm used for the estimation of the function υ in the Jacobian (6). We have considered a nonlinear method without imposing the restriction of linearity considering neural net models.

Hornik et al. [47] showed that any standard feedforward networks with as few as one hidden layer using arbitrary squashing functions are a class of universal aproximators. The results proposed by those authors mentioned above have enabled us to consider a neural network with just one single hidden layer. The number of hidden units (or neurones) in the single hidden layer is determined by statistical methods based on model selection criteria as it appears in the results of this paper. Now let us illustrate how we have obtained a consistent neural net estimator based on the robust estimation of the function υ in the jacobian (Eq. (6)).

Let consider the reconstruction vector $\mathbf{y}_t = \upsilon\left(y_{t-\tau}, y_{t-2\tau}, \dots, y_{t-(m-2)\tau}, y_{t-(m-1)\tau}, y_{t-m\tau}\right)$ as defined by Eq. (5). We obtain the neural network estimator by approximating the unknown non-linear function *upsilon* through a single-layer hidden feedforward network with a single output by

$$\upsilon \approx \hat{\upsilon} = \Phi_0\left[\hat{\alpha}_0 + \sum_{q=1}^{h} \hat{\omega}_{q0} \Phi_q\left(\hat{\alpha}_q + \sum_{j=1}^{m} \hat{\omega}_{jq} y_{t-j\tau}\right)\right] \tag{7}$$

where $\hat{\omega}_{jq}$ are the estimated layers connection weights from hidden layer to output, m is the embedding dimension, $\hat{\alpha}_q$ is the estimated network bias from hidden layer, Φ_q is the transfer function which, in our case, is the logistic function, $\hat{\omega}_{q0}$ are the estimated layers connection weights from input to hidden layer, h is the number of neurones (or nodes) in the single hidden layer, $\hat{\alpha}_0$ is the estimated network bias from input and $\Phi_0 \in I$. The issue of parameter estimation is reduced to the least squares problem in which the quantity to be minimized is defined by

$$\sum_{t=1}^{n}\left(y_t - \left[\alpha_0 + \sum_{q=1}^{h} \omega_{qo} \Phi_q\left(\alpha_q + \sum_{j=1}^{m} \omega_{jq} y_{t-j\tau}\right)\right]\right)^2$$

Then we have got the partial derivates (Eq. (6)) applying the chain rule to Eq. (7) as follows,

$$\frac{\partial \hat{\upsilon}}{\partial y_{t-j\tau}} = \Phi'_0(z_0) \sum_{q=1}^{h} \hat{\omega}_{q0} \Phi'_0(z_q) \hat{\omega}_{jq} \tag{8}$$

where

$$z_0 = \hat{\alpha}_0 + \sum_{q=1}^{h} \hat{\omega}_{q0} \Phi_q(z_q), \quad z_q = \hat{\alpha}_q + \sum_{j=1}^{m} \hat{\omega}_{jq} y_{t-j\tau}$$

and the estimated partial derivatives are provided by

$$\hat{\partial G} = \begin{pmatrix} \frac{\partial\hat{\upsilon}}{\partial y_{t-\tau}} & \frac{\partial\hat{\upsilon}}{\partial y_{t-2\tau}} & \frac{\partial\hat{\upsilon}}{\partial y_{t-3\tau}} & \cdots & \frac{\partial\hat{\upsilon}}{\partial y_{t-(m-1)\tau}} & \frac{\partial\hat{\upsilon}}{\partial y_{t-m\tau}} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{9}$$

Finally we get the neural network estimator of the zth Lyapunov exponent given as

$$\hat{\lambda}_z = \lim_{M\to\infty} \frac{1}{M} \log \mu_z\left(\left|\hat{\partial G}^M\right|\right) \tag{10}$$

where μ_z is the zth largest eigenvalue provided by the jacobian $\hat{\partial G}^M = \hat{\partial G}(y_M) \cdot \hat{\partial G}(y_{M-1}) \cdot \ldots \cdot \hat{\partial G}(y_1)$ for $z = 1, 2, 3, \ldots, m$ where $\hat{\partial G}(\cdot)$ are the partial derivates estimated from the best-fitted neural net model (Eq. (9)), M denotes the block length and m is the embedding dimension.

Shintani and Linton [75] pointed out that in order to ensure that the estimate is consistent, the full sample should not be considered. Instead of that we must divide it into blocks. Since the number of evaluation points is less than or equal to the sample size n, the block length M can be also understood as the sample size of a subsample. Hence the choice of the block length M is an important issue in practice.

Mc Caffrey et al. [63] discussed the optimal choice of the block length suggesting that averaging the Lyapunov exponent estimators from the non-overlapping blocking method ($B = n/M$) might reduce the overall bias, for a review see also [27,65]. However, it should be noted that such an estimate in the one-dimensional case is identical to the estimate based on the full sample ($M = n$).

Whang and Linton [82] illustrated that the Lyapunov exponent estimator can be derived not only from the non-overlapping blocking method but also from any other subsampling method. They proposed the equally spaced blocking method ($B = n/M$). In fact, they showed that the optimal subsample for the Lyapunov exponent estimator depends on the data generating process. Therefore, we may use either the non-overlapping or the equally spaced blocking method as a choice of subsample, even a bootstrap blocking method as argued [71] which takes random samples without replacement by each block B. In this paper we have considered both the full sample and three different methods of subsampling by blocks as we shall see below. Now let us focus on the asymptotic properties of the Lyapunov exponent estimator.

Shintani and Linton [75] provided a statistical framework for testing the hypothesis of chaos (2) based on the neural net estimator of the Lyapunov exponent and the consistent estimator of its variance. The asymptotic properties of the estimator can be described as follows. Let λ_z be the theoretical Lyapunov exponent derived by Eq. (3). Let $\hat{\lambda}_z$ be the non-parametric neural network estimator of the Lyapunov exponent defined by Eq. (10) for a block length M and $z = 1, \ldots, m$ where m denotes the embedding dimension.

Shintani and Linton [75] proved the asymptotic normality of that estimator under certain conditions by

$$\sqrt{M}\left(\hat{\lambda}_z - \lambda_z\right) \sim N(0, \varphi_z) \tag{11}$$

where φ_z is the asymptotic variance of the zth Lyapunov exponent estimator. They showed that $\hat{\varphi}_z$ is a consistent long-term variance estimator of φ_z given by

$$\hat{\varphi}_z \equiv Var\left(\hat{\lambda}_z\right) = \lim_{M\to\infty} Var\left(\frac{1}{\sqrt{M}} \sum_{t=1}^{M} \eta_{z,t}\right) \tag{12}$$

where M is the block length that is equal to n for the full sample. The quadratic spectral kernel function $\eta_{z,t}$ is given by $\eta_{z,t} = \xi_{z,t} - \hat{\lambda}_z$. We estimate those long-term variances using the method proposed by Andrews [3]. The parameter $\xi_{z,t}$ is given by

$$\xi_{z,t} = \frac{1}{2} \log \mu_t\left(\left|\hat{\partial G}^t\right|\right) - \frac{1}{2} \log \mu_{t-1}\left(\left|\hat{\partial G}^{t-1}\right|\right) \tag{13}$$

for $t \geq 2$ and $\xi_{z,1} = \frac{1}{2} \log \mu_t\left(\left|\widehat{\partial G}^t\right|\right)$ where $\hat{\partial G}^t = \hat{\partial G}(y_t) \cdot \hat{\partial G}(y_{t-1}) \cdot \ldots \cdot \hat{\partial G}(y_1)$ and $\hat{\partial G}(\cdot)$ are the partial derivates following Eq. (9) for $t = 1, 2, \ldots, M$.

Then, [75] proposed a one-sided test for testing the hypothesis of chaos (2). This statistical test is given by

$$\hat{t}_z = \frac{\hat{\lambda}_z}{\sqrt{\hat{\varphi}_z/M}} \sim N\left(0, \hat{\varphi}_z\right) \tag{14}$$

So we want to know if the estimated Lyapunov exponent are (or are not) statistically significant greater than 0. Under the null hypothesis H_0: the data-generating process is chaotic, the Lyapunov exponent estimator $\hat{\lambda}_z$ leads to asymptotically valid inferences in that the associated p-value follows a normal distribution on $N\left(0, \hat{\varphi}_z\right)$. Reject the null hypothesis $H_0 : \hat{\lambda}_z > 0$

Table 4
The estimated Lyapunov exponents by each cryptocurrency based on the direct method ($\hat{\lambda}_D$) and the Jacobian indirect method from different blocking methods ($\hat{\lambda}_F$=Full, $\hat{\lambda}_N$=Non-overlapping, $\hat{\lambda}_E$=Equally spaced, $\hat{\lambda}_B$=Bootstrap). Results pertaining to the returns time series from the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) rates over the last 5 years are shown on the left. Those of the squared returns (volatility) are shown on the right.

	$\hat{\lambda}_{Direct}$	$[m,\tau,h]$	$\hat{\lambda}_{Indirect(F)}$	$\hat{\lambda}_{Indirect(N)}$	$\hat{\lambda}_{Indirect(E)}$	$\hat{\lambda}_{Indirect(B)}$	$\hat{\lambda}_{Direct}$	$[m,\tau,h]$	$\hat{\lambda}_{Indirect(F)}$	$\hat{\lambda}_{Indirect(N)}$	$\hat{\lambda}_{Indirect(E)}$	$\hat{\lambda}_{Indirect(B)}$
BTC	−0.11885	[8,1,4]	**0.15347**	**0.16289**	**0.17775**	**0.17987**	−0.23554	[8,1,7]	**0.20745**	**0.21981**	**0.22129**	**0.22539**
			(3.4e-03)*	(2.7e-04)*	(2.4e-04)*	(2.0e-04)*			(2.4e-03)*	(2.7e-04)*	(3.4e-04)*	(2.0e-04)*
ETH	−0.34846	[7,1,7]	**0.07228**	**0.08462**	**0.09491**	**0.10462**	−0.64181	[7,1,2]	**0.16469**	**0.17787**	**0.17912**	**0.18314**
			(1.4e-03)*	(4.3e-04)*	(4.4e-04)*	(4.3e-04)*			(1.4e-03)*	(2.3e-03)*	(4.0e-03)*	(4.6e-03)*
LITE	−0.39857	[8,1,9]	**0.19094**	**0.20275**	**0.21643**	**0.22731**	−0.48881	[9,1,10]	**0.25219**	**0.26035**	**0.27891**	**0.27995**
			(2.1e-03)*	(4.1e-04)*	(4.2e-04)*	(4.0e-04)*			(8.0e-04)*	(1.7e-03)*	(3.4e-03)*	(3.1e-03)*
XRP	−0.21034	[9,1,10]	−0.00542	**0.01516**	**0.01763**	**0.01244**	−0.39217	[10,1,3]	−4.7e-05	**0.00681**	**0.00527**	**0.00504**
			(3.8e-03)	(5.4e-02)	(5.3e-02)	(5.2e-02)			(6.7e-04)	(4.3e-03)	(4.2e-03)	(4.9e-03)

means lack of chaotic behaviour. Remember that the asymptotic distribution of the estimator obtained from direct methods do not exist, that is, there is not chance to make statistical inference about the direct Lyapunov exponent estimator and test if the estimated values are or not statistically significant greater than 0.

Now we are going to report the main results of this section. We have used the top 4 cryptocurrencies for testing our proposal related to the chaos model-data paradox in this context. That is, chaos is elusive in cryptocurrency time series data because normally a correct and robust method to detect it is not used in this field. For this purpose we have compared the results obtained by the traditional direct methods and the Jacobian indirect methods. We have followed the novel R package called *DChaos* developed by Sandubete and Escot [71] which contains the algorithms that we have considered in this paper regarding the Jacobian indirect methods.

These algorithms are publicly available at CRAN repository www.CRAN.R-project.org/package=DChaos. We have also considered the traditional direct methods proposed by the *tseriesChaos* package provided by Fabio [2019]. This last R package is based on ideas suggested by Hegger et al. [46]. The *tseriesChaos* package implements the direct method provided by Kantz [51] and it is also available at CRAN repository.

We have set the following parameters to save time in computational operations: embedding dimension $3 \leqslant m \leqslant 10$, time-delay $1 \leqslant \tau \leqslant 10$, number of nodes in the single hidden layer $2 \leqslant h \leqslant 10$. The R packages which implement the traditional direct method use heuristic approaches for estimating the embedding parameters as stated above. The Jacobian indirect methods provided by the DChaos package considers by default the statistical approach based on model selection procedures. In this case, we have applied the BIC together with the leave one out cross-validation technique.

Once the coefficients have been estimated, the residuals are extracted and the BIC criteria is applied as follows:

$$BIC = \log(RSS) + \frac{\log(n)}{n}[1 + coef(m+2)]$$

where *RSS* is the residual sum of squares, n is the number of observations, m is the embedding dimension and $coef$ is the number of coefficients considered in each nonlinear regression as noted above. We have estimated 720 different neural nets models from each cryptocurrency time-series data considered (5760 regression models) in order to get the results shown in Table 4.

The estimated Lyapunov exponents by each cryptocurrency based on the direct method are denoted by $\hat{\lambda}_D$. We have considered following [71] the full sample ($\hat{\lambda}_F$) and three different blocking methods: non-overlapping subsampling ($\hat{\lambda}_N$), equally spaced subsampling ($\hat{\lambda}_E$) and bootstrap subsampling ($\hat{\lambda}_B$) when applying the Jacobian indirect methods.

The Lyapunov exponents corresponding to the best-fitted neural net model (with a lower BIC value) for each blocking method is shown. Standard errors are written in parentheses and the optimal parameter set $[m,\tau,h]$ are denoted into brackets. We have remarked in bold those Lyapunov exponents with positive values and with asterisk $*$ those which are statistically significant at the 99% confidence level. Median values of all used blocks are presented for the estimation based on blocking methods. The block length M has been chosen following [75] as $M = int\left[c \times (n/\log n)^{1/6}\right]$ with $c = 36.2$. QS Kernel with optimal bandwidth has been used for the heteroskedasticity and autocorrelation consistent covariance matrix estimation following [3].

The data shown in Table 4 provide the following comments. First, we have got only positive Lyapunov exponents by indirect methods. All values obtained from direct methods are negative. This fact could provide us a potencial solution to the chaos model-data paradox in the cryptocurrency market because most recently published studies tend to apply the direct methods instead of the Jacobian indirect methods.

Second, if we consider a a sufficiently high value of the embedding dimension m (above 7) the estimated Lyapunov exponents are positive and significant in almost all cryptocurrency rates considered: Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), considering both return and volatility time series. The Ripple (XRP) rate were positive but not significant in both cases. In fact, it is the only one that shows negative values. Therefore, these results provide clear evidence of chaotic behaviour in almost all cryptocurrency rates and periods considered.

Third, the optimum time-delay is $\tau = 1$ in all cases as [78] suggested. Fourth, the results given by the blocking methods are better than the ones provided by the full sample. The bootstrap method gives better results on average than other

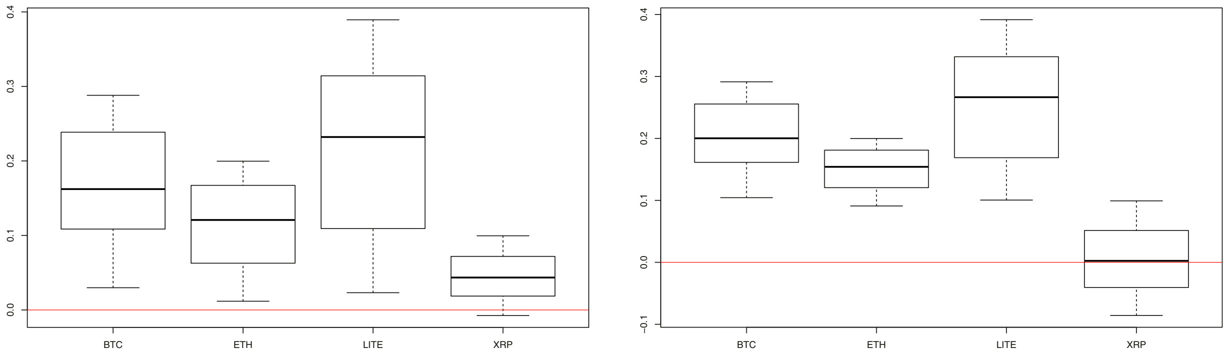

Fig. 4. Box-plot from the Lyapunov exponent based on the bootstrap blocking method. We have considered 720 different neural nets models estimated by each cryptocurrency for $3 \leqslant m \leqslant 10$, $1 \leqslant \tau \leqslant 10$ and $2 \leqslant Q \leqslant 10$. Results pertaining to the returns time series from the Bitcoin (BTC), Ethereum (ETH), Litecoin (LITE), and Ripple (XRP) rates over the last 5 years are shown on the left. Those of the squared returns (volatility) are shown on the right.

blocking methods as illustrated by Sandubete and Escot [71]. These facts are consistent with the suggestion proposed by Shintani and Linton [75] on the use of blocking methods instead of full sampling when estimating the Lyapunov exponents.

Finally we have illustrated the box-plots in Fig. 4 considering the largest Lyapunov exponents got from all the neural net models by each cryptocurrency rate based on the bootstrap blocking method. The number of bootstrap iterations is $B = 1000$. Results pertaining to the returns are shown on the left. Those of the squared returns (volatility) are shown on the right.

The estimated median value (bold line) of the Lyapunov exponents from 720 different neural net regressions are positives in all the cryptocurrency rates considering both return and volatility time series only the Ripple (XRP) rate reports negative values of all estimated Lyapunov exponents, see Fig. 4. To sum up, we have been able to detect chaotic signals inside some real-world financial time-series data, at least in the four major cryptocurrency rates considered and during the periods chosen.

5. Conclusion

We conclude by stating that we have been able to detect chaos directly from actual financial data based on the Bitcoin, Ethereum, Ripple and Litecoin cryptocurrencies rates. These results reject the efficient markets hypothesis and provide a potencial solution to the chaos model-data paradox. Nevertheless, in this paper we have not attempted to generalise this conclusion to all financial series or even to all cryptocurrency time series.

The main interest of this paper has been to show that robust evidence of nonlinearities and chaos can emerge from financial real time series when using new computational robust methods based on statistical approaches. At least, in the 8 cryptocurrency rates and periods considered from the returns and volatility time series. These facts could open up a new line of research in which new contributions could appear considering other financial assets, periods and/or methods to estimate the Lyapunov exponents in order to detect chaotic signals inside financial time series.

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Declaration of Competing Interest

No conflict of interest exists.

CRediT authorship contribution statement

Lukasz Pietrych: Conceptualization, Methodology, Data curation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Visualization. **Julio E. Sandubete:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Visualization. **Lorenzo Escot:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Supervision, Project administration, Funding acquisition.

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