Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 3586012, 9 pages
https://doi.org/10.1155/2019/3586012

Research Article

An Analytic Expression for the Inverse Involute

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Received 16 June 2019; Accepted 1 August 2019; Published 29 September 2019

Academic Editor: Jean Jacques Loiseau

This article introduces new types of rational approximations of the inverse involute function, widely used in gear engineering, allowing the processing of this function with a very low error. This approximated function is appropriate for engineering applications, with a much reduced number of operations than previous formulae in the existing literature, and a very efficient computation. The proposed expressions avoid the use of iterative methods. The theoretical foundations of the approximation theory of rational functions, the Chebyshev and Jacobi polynomials that allow these approximations to be obtained, are presented in this work, and an adaptation of the Remez algorithm is also provided, which gets a null error at the origin. This way, approximations in ranges or degrees different from those presented here can be obtained. A rational approximation of the direct involute function is computed, which avoids the computation of the tangent function. Finally, the direct polar equation of the circle involute curve is approximated with some application examples.

1. Introduction

The circle involute function, used to mathematically describe the flank of the gear tooth, is formulated as $x = \mathrm{inv}(u) = \tan(u) - u$ (see Figure 1), where u is the pressure angle and x is the polar angle. On the other hand, the inverse involute function that computes the angle from the involute is used, for example, in the computation of the center distances between the gear axes, and cannot be expressed mathematically in a direct way. So, either tables or the best approximate formulae obtained to date should be used (see Section 2). The shape of the inverse involute function, until 70° (1.5 rad approx.), for example (see Figure 2), shows an elbow in its initial part, starting at the origin with a very high value in its derivative and then changing until almost horizontal. It is known that rational functions are especially appropriate to approximate functions with that elbow shape (see [1]). The authors of this article worked previously with the applications of the approximation theory by rational functions applied to tyre models, with elbow-shaped curves similar to the inverse involute, and obtained very efficient and accurate approximation results (see [2–6]). This article first presents information from previous works and then develops the theoretical foundation and algorithms used to obtain the approximate expressions of both the inverse involute and direct functions proposed at the end.

2. Previous Works

Several authors have been publishing formulations that compute the inverse involute function. In the following sections, we will call x the involute of an angle u(rad): $x = \mathrm{inv}(u)$; $u = \mathrm{inv} - 1(x)$. To compare the errors, we will take as a reference a maximum value of the angle $u = 45^\circ = 0.785398\,\mathrm{rad}$, which has an involute value of $x = 0.2146$. This value fulfills a major part of the engineering application requirements.

Initially, we must cite the work of Dudley [7], who proposed the following expression:

$$u \simeq 1.441 \cdot x^{1/3} - 0.366 \cdot x, \tag{1}$$

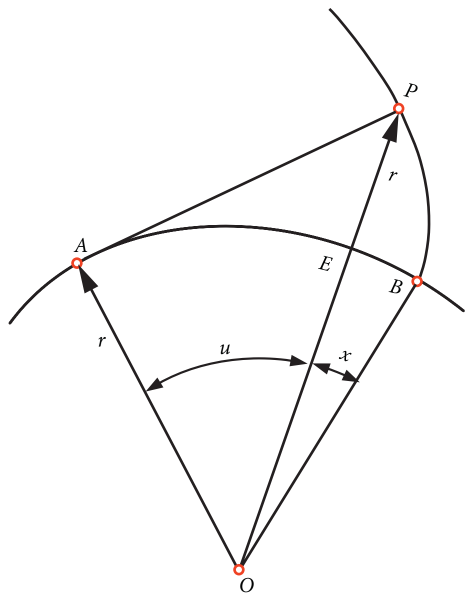

Figure 1: The circle involute.

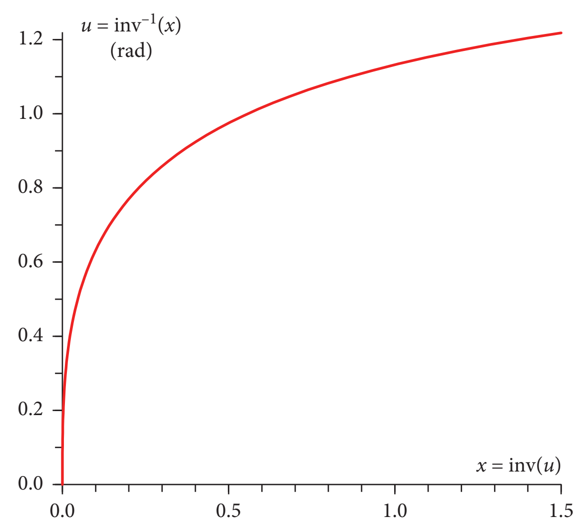

Figure 2: The inverse involute function.

which is valid for $x \leq 0.5$ with a maximum error for $u = 45^\circ$ and $E_{max} = 0.0012$ rad $= 4.12529'$. This error has been calculated when comparing it with equation (3). Due to its error, this expression is only valid as a first approximation.

The iterative iteration by Laskin is far more accurate, which is as follows [8]:

$$\begin{aligned} u_1 &\simeq 1.441 \cdot x^{1/3} - 0.374 \cdot x, \\ u_2 &\simeq u_1 + \frac{(x - \mathrm{inv}(u_1))}{(\tan(u_1))^2}, \\ u_3 &\simeq u_2 + \frac{(x - \mathrm{inv}(u_2))}{(\tan(u_2))^2}. \end{aligned} \tag{2}$$

This iterative formulation is valid for angles up to 65°. Its error grows with x, and it takes values for $u = 45^\circ$ and $E_{max} = 0.000017$ rad $= 3.506''$ with two iterations (computation until u_2). This error has been calculated when comparing it with equation (3). Its error is reduced when the number of iterations increases.

On the other hand, Cheng [9] used the perturbation theory, implementing the method of dominant balance in order to obtain approximations to the inverse involute function, proposing the following expression:

$$\begin{aligned} u \simeq{} & 3^{1/3} \cdot x^{1/3} - \frac{2}{5} \cdot x + \frac{9}{175} \cdot 3^{2/3} \cdot x^{5/3} - \frac{2}{175} \cdot 3^{1/3} \cdot x^{7/3} \\ & - \frac{144}{67375} \cdot x^{9/3} + \frac{3258}{3128125} \cdot 3^{2/3} \cdot x^{11/3} \\ & - \frac{49711}{153278125} 3^{1/3} \cdot x^{13/3} - \frac{1130112}{9306171875} \cdot x^{15/3} \\ & + \frac{5169659643}{95304506171875} 3^{2/3} \cdot x^{17/3}. \end{aligned} \tag{3}$$

The previous expression in these 9 terms has a very low error, which takes values for $u = 45^\circ$ and $E_{max} = 1.58E-9$ rad $= 3.2589E-4''$ (see [10]). For its computation, it requires 38 basic floating-point operations (FLOPs) plus a cube root.

In his article [9], Cheng economized (see Section 3.1) the previous truncated series while taking into account only the first 4 terms (until $x^{7/3}$) when performing the transformation $t = x^{2/3}$, so the resulting series is as follows:

$$u \simeq t^{1/2} \cdot \left(3^{1/3} \cdot - \frac{2}{5} \cdot t + \frac{9}{175} \cdot 3^{2/3} \cdot t^2 - \frac{2}{175} \cdot 3^{1/3} \cdot t^3 \right). \tag{4}$$

As the term "$t^{1/2}$" multiplies the brackets, the result will pass through the origin. Then, the 3-degree series within the brackets is economized until 1 degree using the Chebyshev polynomials, undoing the change afterwards, $x = t^{3/2}$, so that the following equation is obtained:

$$u \simeq 1.440859 \cdot x^{1/3} - 0.3660584 \cdot x. \tag{5}$$

Very similar to calculation by Dudley, valid for $0 \leq u \leq 45^\circ$, with a maximum error $E_{max} = -0.001308$ rad $= 4.49979'$. When the angle is 45°, the value of $(\mathrm{Inv}(x))^{1/3}$ is 0.5987026.

In 2017, more recent works have been published by Liu [11], using a method similar to the one used by Cheng, but applying the integral expression of the arctan function

$$u = \arctan(c) = \int_0^y \frac{1}{1 + c^2} d, \tag{6}$$

where c is an intermediate variable ($c = \tan(u)$). Then, Liu expanded the tangent function in Taylor Series, obtaining finally the following two expressions: the first one, more simple but less accurate, is similar to Cheng's (3):

$$\begin{aligned} u \simeq{} & (3 \cdot x)^{1/3} - \frac{2}{15} \cdot (3 \cdot x)^{3/3} + \frac{3}{175} \cdot (3 \cdot x)^{5/3} \\ & - \frac{2}{1575} \cdot (3 \cdot x)^{7/3} + \frac{528}{6670125} \cdot (3 \cdot x)^{9/3} \\ & - \frac{362}{9384375} \cdot (3 \cdot x)^{11/3}, \end{aligned} \tag{7}$$

with an error (at $u = 45^{\circ}$) $E_{\max} = 2.75E-5$ rad $= 1.73E-3''$. The error continues growing for bigger values of u. The second expression, which is very accurate, but not too simple to compute, proposed in Liu's work is

$$u \simeq \arccos\left(\frac{\sin\left(\arctan\left((3\cdot x)^{1/3} + (3/5)x + (1/11)x^{8/3}\right)\right)}{x + \arctan\left((3\cdot x)^{1/3} + (3/5)x + (1/11)x^{8/5}\right)}\right), \tag{8}$$

with an error (at $u = 45^{\circ}$) $E_{\max} = 1.73E-11$ rad $= 3.57E-6''$ (see [11] for more details).

In this article, we start from Cheng's expression (3) but we obtain approximations truncating expansions in a series of Jacobi polynomials and a rational series; in order to get that, we start from the Chebyshev–Padé rational approximations (see [12]) and the original formulation of the Remez algorithm, which is modified thus imposing the condition of a null error at the origin. The resulting expressions are very compact and quickly executed, with a very low error. To justify the resulting final expressions and compute some others, the concepts of expansion in a series of the Chebyshev polynomials, economization, and approximation of functions are briefly reviewed, and the series of Jacobi polynomials allowing for the selection of areas with a lower error in a flexible way are also described. The theory of rational functions, the Remez algorithm, and its adaptation proposed in this article are reviewed. Finally, the proposed formulae are presented.

3. Theoretical Basis of the Approximations Used in This Article

3.1. Approximation of a Function in the Chebyshev Series. The Chebyshev polynomials (see [10]) of the first kind are defined by $T_n(v) = \cos[n\arccos(v)]$, and they are orthogonal regarding the function $w(v) = (1-v^2)^{-1/2}$ in the interval $[-1, 1]$. For a detailed explanation of the concepts of orthogonality, orthogonal polynomials, and orthogonality intervals, see [10]. To work in different $[a, b]$ intervals, shifted polynomials with the following change must be used:

$$x = \frac{1}{2}\left[(b-a)v + a + b\right]. \tag{9}$$

Successive terms of the Chebyshev polynomials are as follows:

$$\begin{aligned} T_0(v) &= 1, \\ T_1(v) &= v, \\ T_2(v) &= 2v^2 - 1, \\ T_3(v) &= 4v^3 - 3v, \\ T_4(v) &= 8v^4 - 8v^2 + 1. \end{aligned} \tag{10}$$

In [10], we can find formulae to calculate products, integrals, and derivatives of the Chebyshev polynomials. The expansion of a function in the Chebyshev series has the following form:

$$f(v) = \sum_{n=0}^{\infty}{}' a_n T_n(v) \approx \sum_{n=0}^{N}{}' a_n T_n(v), \tag{11}$$

in which the prime symbol in the summation indicates that the first term must be divided by 2, being

$$a_n = \frac{1}{r_n}\int_{-1}^{1} w(v)\cdot f(v) T_n(v)\,dv, \tag{12}$$

where $w(v)$ is the weight function:

$$w(v) = \left(1 - v^2\right)^{-1/2}. \tag{13}$$

As N grows, the series summation approaches to $f(v)$; that is, it converges to $f(v)$. If we truncate the series in degree N, we get an approximation to the function, and it is more accurate with a higher N. Due to the properties of the Chebyshev polynomials, truncating in $N-1$ is the best $N-1$ degree polynomial approximation to the development at N degree, always working in the orthogonality interval $[-1, 1]$; this is called economization (see Burden and Faires [13]) and is the method used by Cheng to obtain equation (5) from equation (4). r_n is the norm of the function ($\pi/2$ for the Chebyshev polynomials).

In the following sections, Jacobi polynomials are introduced because they allow a flexible error choice in some areas of the curve, which will be useful. Then, we will present approximations in rational functions because they are more accurate and easier to compute.

3.2. Expansion in a Series of Jacobi Polynomials. Jacobi polynomials come from the group of classic orthogonal polynomials obtained from the Sturm–Liouville differential equation, from which the Chebyshev polynomials are also obtained (see [10]). The expansion of a function in a series of Jacobi polynomials uses a Jacobi weight function this time:

$$\begin{aligned} f(v) &\approx \sum_{n=0}^{N}{}' a_n \cdot J_n(v), \\ a_n &= \frac{1}{r_n}\int_{-1}^{1} w(v)\cdot f(v) J_n(v)\,dv. \end{aligned} \tag{14}$$

The weight function in Jacobi orthogonal polynomials is as follows:

$$w(v) = (1-v)^{\delta}(1+v)^{\gamma}. \tag{15}$$

This function is controlled by two parameters, δ and γ, that allow choosing the area of a best approximation at the orthogonality interval. The minimum value of both δ and γ is -1. If $\delta = \gamma = -0.5$, then we obtain the weight function of the Chebyshev polynomials (13). If both are equal, $\delta = \gamma$, the error is symmetric regarding the y-axis. If $\delta = \gamma = -1$, then we get a null error at both ends of the interval. If $\delta = -1$, then the error is null at the right end. If $\gamma = -1$, then the error is null at the left side of the interval.

3.3. Approximation in a Jacobi Series of the Inverse Involute Function. We can expand the complete $f(x)$ polynomial obtained by Cheng (see (3)) in a Jacobi series with $\delta = \gamma = -1$, as it is shown in (14), which gives a near-zero error at both ends; however, we have to previously convert equation (3) to the y domain by means of the transformation $y = x^{1/3}$, obtaining a degree 17 polynomial in y. Then, we convert this polynomial from the y domain to the v domain [−1, 1] using (9). After that, integral (14) must be used to calculate the coefficients of the Jacobi series in terms of v. Then, we can truncate the calculation at the desired degree. Finally, we have to undo the previous transformations to go back to the y domain and then to the x domain to obtain the following approximation:

$$u \simeq 1.447492 \cdot x^{1/3} - 0.0472447 \cdot x^{2/3} - 0.29949 \cdot x. \quad (16)$$

It requires 4 products and 2 sums and has an error distributed along the interval with a maximum value $E_{\max} = 0.000222\ \text{rad} = 0.7631'$ for 45°, much smaller than expression (5). Figure 3 shows error curves of both equations (5) and (16).

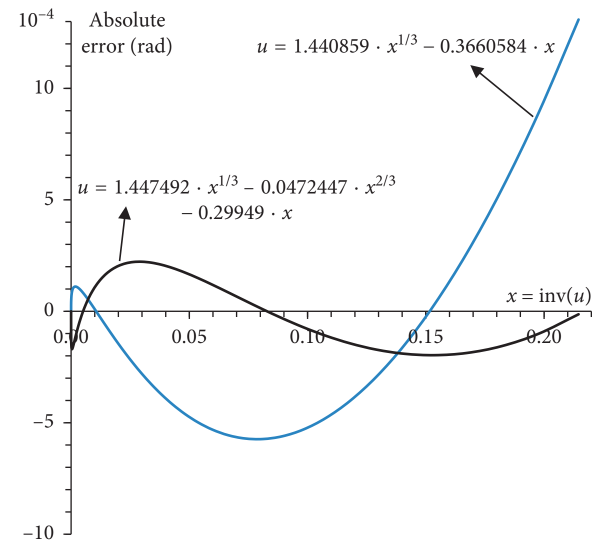

Figure 3: Absolute error in the inverse involute function (approximations (5) and (16)).

3.4. Approximation Using Rational Functions (RA). Approximations in rational functions (see [13]) with polynomials of degree n at the numerator and degree m at the denominator (we will denote them as $[n, m]$) are more efficient when the function varies in a quick way in some areas but not in some others [1], as happens in the inverse involute function. As an example, in [14], we can find a nice application of rational functions to the vibrational frequency of a one-dimensional bar. In [13], we can see the theory of Pade and Chebyshev–Pade rational approximations. In MAPLE, the Chebpade function implements the rational approximations and it allows the numerical conversion between normal polynomials and Chebyshev polynomials too. The Chebyshev–Padé functions obtain good approximations, but not those of the minimum value of the maximum error (known as minimax). To find the latter, the iterative Remez algorithm is used [10, 12, 15–17], which in turn stems from the Chebyshev–Padé approximation; it fine tunes the result by numeric iterations and converges to an improved minimax approximation of both rational and polynomial functions.

According to Tawfik [17], supposing an interval $[a, b]$, Chebyshev's criterion proposes that if $P_n(x)$ is the minimax polynomial of an n degree, there must be at least $(n+2)$ points in this interval where the error function $E(x)$ obtains the maximum absolute value $E_{\max}$ with an alternating sign. Chebyshev demonstrated the existence and unicity of that minimax polynomial $P_n(x)$ (or rational function $R_n(x_i)$). This criterion is mathematically defined with the following equations:

$$\begin{gathered} a \le x_0 < x_1 < \cdots < x_{n+1} \le b, \\ f(x_i) - P_n(x_i) = (-1)^i E_{\max}; \quad i = 0, 1, \ldots, n+1, \\ E_{\max} = \pm \max_{a \le x \le b} \left| f(x) - P_n(x) \right|. \end{gathered} \quad (17)$$

In this manner, we can obtain a system of $(n+2)$ equations with $(n+2)$ unknown (the coefficients of $P_n(x)$, or $R(x)$, and $E_{\max}$). The system will be nonlinear in the case of a rational approximant $R_n(x)$. Solving it, we can get a new set of points x_i with alternating the maximum error $E_{\max}$, which will be more reduced in this situation. Repeating this iteration several times, we can reach the Chebyshev minimax condition so that the value of the error function in the final set of $(n+2)$ points ($E_{\max}$) represents the absolute maximum value of the approximate error (see [17] for more details).

Sometimes imposed constraints are needed for the approximating function, for example, when the original function takes a null value at the origin and it seems conceptually unacceptable to take a nonzero value at the origin in the approximated function. In this paper, the previously developed Remez algorithm has been adapted to obtain rational approximations with null error at the origin just adding that mathematical constraint to the general algorithm presented in [17]:

$$\begin{gathered} f(x_i) - R_n(x_i) = (-1)^i E_{\max}; \quad i = 1, 2, \ldots, n+1, \\ f(x_0) = R_n(x_0). \end{gathered} \quad (18)$$

To do this, the iterations can start with the solution achieved through Jacobi polynomials in the case $\delta = -0.5$ and $\gamma = -1$, which gives a null error at the origin; however, minimax is not taken as the local maxima of the error function, which take different values. In summary,

(1) We started from the Cheng 17-degree polynomial (see equation (3)).

(2) We convert it into a rational function [1, 2] using the Chebyshev–Padé method (see [13]). Previously, the result has been transformed to the y domain.

(3) Then, we apply the adapted Remez algorithm, starting the iteration with the resulting roots in the previous step, imposing the constraint of null error at the origin.

We will apply this 3-step method to the inverse involute function in Section 4 to obtain Apsol8.

To either manipulate the polynomials or approximate them to rational functions and vice versa, the direct and inverse Chebyshev–Padé method can be used (see [13]).

Applications of expansions in a Chebyshev and Jacobi series of rational functions to special functions, heating and air conditioning, vehicle dynamics, steering mechanisms, and tyre nonlinear behavior can be found in [18].

4. Rational Approximation to the Inverse Involute Function by the Adapted Remez Algorithm

Once we have the required theoretical background, we apply the method described in Section 3.4 to obtain Remez minimax approximations passing through the origin for the inverse involute function from the Cheng polynomial (3) and obtain the following:

$$\text{Apsol1} = \frac{-0.0000044041 + 1.3904457432 \cdot x^{1/3}}{0.963946704 + (0.0007905044 + 0.2664596767 \cdot x^{1/3}) \cdot x^{1/3}}. \tag{19}$$

We could see graphically regarding the 17-degree polynomial that this solution has a maximum error of $8.34E-6$ rad ($1.72''$) and the error at the origin is not null.

Applying the Remez algorithm with the null error at the origin constraint, the following can be obtained:

$$\text{Apsol2} = \frac{x^{1/3}}{0.6932757402 + (0.0005653682 + 0.1916191427 \cdot x^{1/3}) \cdot x^{1/3}}. \tag{20}$$

In Figures 4 and 5, we can see the errors of the approximations Apsol 1 and Apsol2. Apsol 1 is obtained using the Chebyshev–Padé algorithm, it does not pass at the origin as shown more clearly in Figure 5. This process has a maximum error of $E_{\max} = 7.5E-6$ rad ($1.54''$) and null error at the origin, and it requires two operations less than the previous process (Apsol1) to include, 5 FLOPs in total plus the cube root.

If we want to reduce the number of decimals, admitting bigger errors, the approximation can be

$$\text{Apsol3} = \frac{x^{1/3}}{0.69328 + (0.000565 + 0.1916 \cdot x^{1/3}) \cdot x^{1/3}}. \tag{21}$$

Graphically, we could see that a slightly bigger maximum error ($E_{\max} = 8.847E-6\ \text{rad} = 1.82''$) occurs but also passes through the origin and requires 5 FLOPs plus the cube root.

As the value of the standard pressure angle is 20°, we will frequently need approximations in its environment; therefore, if we consider the interval between 0° and 30°, approximations will be either more accurate or simple. If we repeat the previous process, we can obtain

$$\text{Apsol4} = \frac{x^{1/3}}{0.69336473 + (-0.0000654976 + 0.1926063 \cdot x^{1/3}) \cdot x^{1/3}}, \tag{22}$$

which passes through the origin and has a maximum error $E_{\max} = 1.124E-7\ \text{rad} = 0.0231''$ in the interval (0°–30°).

In the previous Apsol4, the term $x^{1/3}$ is negative and is almost null, but for the range 0°–45° (Apsol2), the term is positive, indicating that it achieves a value very close to 0 at some intermediate point between 30° and 45°. This error is observed to be almost zero in the interval 0°–35°, so it can be rounded with a very low error, obtaining a very compact and very accurate final expression:

$$\text{Apsol5} = \frac{x^{1/3}}{0.693357 + 0.192484 \cdot x^{2/3}}. \tag{23}$$

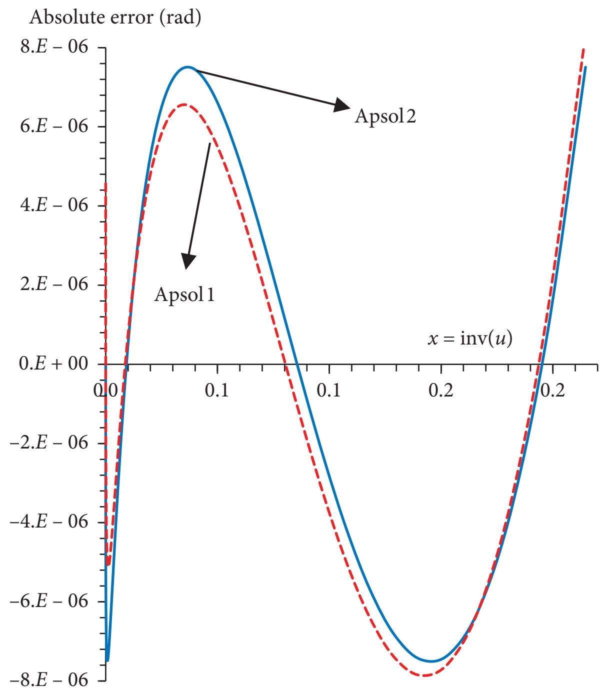

Figure 4: Absolute error in the approximations Apsol1 and Apsol2 of the inverse involute function.

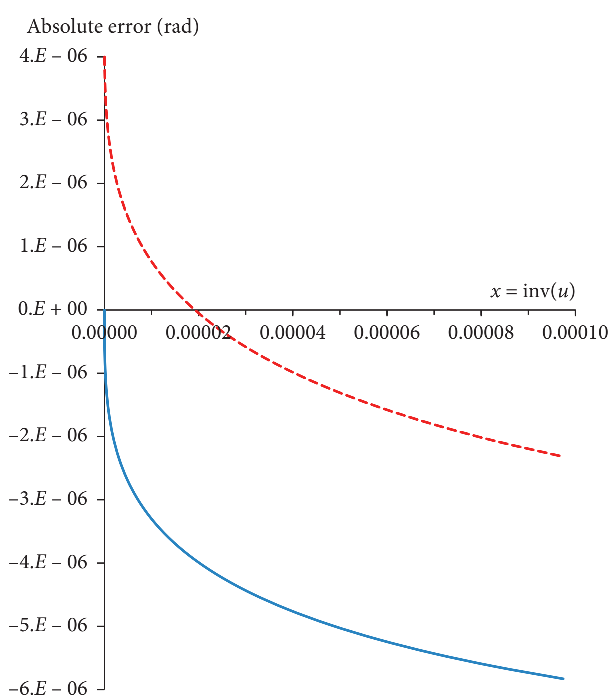

Figure 5: Zoomed-in view of Figure 4 near the origin.

This also passes through the origin and has a very low maximum error $E_{max} = 6.6E - 7\,\text{rad} = 0.137''$ in the interval (0°–35°), requiring only 3 FLOPs plus the cube root.

The limit of 35° reaches the head radius with $z \leq 14$. The limit of 45° reaches the head radius with $z \leq 7$, in which z is the number of teeth.

Table 1 shows a summary of the approximations to the inverse involute function presented in this article.

The three new formulae have been designed to work within the intervals shown at column 2: 0°–45° for the first two formulae and 0°–35° for the latter. The values of maximum errors can be found within the intervals. If they are used outside these intervals, the errors grow very fast as we can see in Figures 3 and 4, so they cannot be used at all.

The interval 0°–35° is wide enough for many engineering applications, if we work in that interval, the third expression is very compact, easy to use, and more accurate than the others.

Liu's rational inverse trigonometric function (8) and Liu's 6 terms (7) are more accurate, but are far more complicated. Dudley's, Laskin's, and Cheng's polynomials (2 terms) are less accurate. For these reasons, we recommend the third new formula Apsol5.

5. Rational Approximation of the Direct Involute Function

The approximation of the inverse involute using polynomial (3) and the rational approximations obtained from equation (3) have the drawback of computing of the cube root of x, which is not very computationally efficient. However, the development in series of that same inverse involute function in terms of x, which should be more efficient, requires polynomials of a higher degree or in the case of rational functions higher degrees at the numerator and the denominator.

The exact computation of the involute function can be directly performed, $\text{inv}(u) = \tan(u) - u$, which is why its polynomial or rational approximation is less interesting. Nevertheless, as the computation of the function $\tan(u)$ is not very efficient and in order to complete this work, this approximation is presented as another example of the application of the methods described here. Rational expressions in terms of u can be obtained with a reasonable approximation for the direct involute function, with lower degree polynomials than in the inverse involute, as shown below with a [2, 2] degree example:

$$\text{Inv}(u) \simeq \frac{(-0.0001653167 + 0.0000297695 \cdot u) \cdot u}{1 - (0.02569798 - 0.00019902 \cdot u) \cdot u}. \tag{24}$$

It is graphically shown that the maximum error up to 45° is as follows: $E_{max} = 0.0003209$, 0.15% of the full-scale value (0.2146). It requires 8 FLOPs but avoids the calculation of the tangent. Obviously, by inverting Apsol5, we could obtain another approximation to the direct involute as follows:

$\text{Inv}(u) \simeq 17.528\left((1/u) - \sqrt{(1/u^2) - 0.5338408}\right)^3$, which is valid up to 35° with an ultralow maximum error in this interval: $E_{max} = 8.37E - 11$, $9.36E - 8\%$ of the full-scale value (0.089342).

If we use Apsol3, we can obtain the following:

Table 1: Summary of the approximations to the inverse involute function.

Formula	Max error (rad) Range	FLOP CR = cube root
Existing in the literature		
Dudley $u = 1.441 \cdot x^{1/3} - 0.366 \cdot x$	$1.2E-3$ rad ($4.12529'$) Range = 0°–45°	3 + CR
Laskin (2 iterations)	$1.7E-5$ rad = $3.506''$ Range = 0°–45°	Uses trigonometric functions
Cheng (9 terms): equation (3)	$1.58E-9$ rad = $3.2589E-4''$ Range = 0°–45°	38 + CR
Cheng (2 terms) (economization): $u \simeq 1.440859 \cdot x^{1/3} - 0.3660584 \cdot x$	$1.308E-3$ rad = $4.4998'$ Range = 0°–45°	3 + CR
Liu (6 terms)	$2.75E-5$ rad = $5.68''$ Range = 0°–45°	21 + CR
Liu (rational inverse trigonometric function): $u \simeq \arccos(\sin(\arctan((3\cdot x)^{1/3} + (3/5)x + (1/11)x^{8/5}))/x + \arctan((3\cdot x)^{1/3} + (3/5)x + (1/11)x^{8/5}))$	$1.09E-9$ rad = $2.261E-4''$ Range = 0°–45°	Uses inverse trigonometric functions
New formulae		
$u \simeq 1.447492 \cdot x^{1/3} - 0.0472447 \cdot x^{2/3} - 0.29949 \cdot x$	$2.22E-4$ rad = $0.763'$ Range = 0°–45°	6 + CR
$u \simeq x^{1/3}/0.69328 + (0.000565 + 0.1916.x^{1/3}) \cdot x^{1/3}$	$8.847E-6$ rad = $1.82''$ Range = 0°–45°	5 + CR
$u \simeq x^{1/3}/0.693357 + 0.192484 \cdot x^{2/3}$	$6.6E-7$ rad = $0.137''$ Range = 0°–35°	3 + CR

$$\operatorname{Inv}(u) \simeq \frac{0.130483 \cdot \left(-0.0113 \cdot u + 20 + \sqrt{400 - 212.5325 \cdot u^2 - 0.452 \cdot u}\right)}{u}, \quad (25)$$

which is valid up to 45° with a maximum error $E_{\max} = 6.35E-8$, $2.95E-5\%$ of the full-scale value (0.2146).

6. Applications

The proposed expressions can simplify many calculations in gear engineering. We show here some examples:

6.1. The Direct Polar Equation of the Involute Curve. As we know from Figure 1, the polar equation of the circle involute curve at a given radius r_x is $x = \operatorname{Inv}(u) = \tan(u) - u$; $u = \operatorname{Inv}^{-1}(x)$. Assuming that we use the standard pressure angle of 20°, if r_x is the pitch radius ($r = r_x$), then $u = 20°$ and

$$r_x = r \cdot \frac{\cos(20°)}{\cos(u)} = r \cdot \frac{\cos(20°)}{\cos(\operatorname{Inv}^{-1}(x))}. \quad (26)$$

We can use the previous techniques to obtain a direct relation between the radius r_x and the polar angle x. In this case, the approximate function does not take the value 0 at the origin and a zero error does not need to be forced at the origin. We can find the following minimax rational [2, 2] expression, assuming a standard pressure angle of 20°:

$$r_x \approx r \cdot \frac{0.9396884 - 0.0375577 \cdot x^{1/3} + 0.7518366 \cdot x^{2/3}}{1 - 0.04043 \cdot x^{1/3} - 0.2323572 \cdot x^{2/3}}. \quad (27)$$

This result is valid up to $u \leq 35°$. It has a low maximum error. In a function of the pitch radius, we will have approximately $E_{\max} = (r/25)E-4$ (obtained by calculating the error for different values of r), both the pitch radius and the error being expressed in mm.

Of course, we could directly use the Apsol5, which would be much more accurate, assuming the following computation of cosine functions:

$$r_x \approx r \cdot \frac{\cos(20°)}{\cos(x^{1/3}/(0.693357 + 0.192483 \cdot x^{2/3}))}. \quad (28)$$

The result is valid up to $u \leq 35°$ with a maximum error $E_{\max} = (r/25)E-6$, in which both the pitch radius and the error are expressed in mm. In the range up to $u \leq 45°$, we can use Apsol3 to obtain a maximum error $E_{\max} = (r/10)E-5$, with the pitch radius and the error expressed in mm.

Similar expressions could be obtained for different standard pressure angles (25°, for example).

6.2. Center Distance When Using Shifted Gears with Zero Circular Tolerance. In this case, it is known that the functioning pitch radii are

$$r' = r \cdot \frac{\cos(20°)}{\cos(u')},$$
$$\operatorname{Inv}(u') = 2\frac{s_1 + s_2}{z_1 + z_2} tg(20) + EV(20), \quad (29)$$

in which u' is the operating pressure angle that gives 0 circular tolerance and good transmission, when using toothed wheels with z_1 and z_2 teeth and s_1 and s_2 as the profile shifts. From the calculation of $\mathrm{Inv}(u')$, we can use Apsol8 to obtain the pitch radii r' and the center distance.

The polar equation of the circle involute curve can be used to simplify calculations in many applications, as mentioned in previous research [19], where the authors calculate the load distribution along the line of contact for involute external gears and the elastic potential energy of a spur tooth, or the approximate method shown in another study [20].

7. Conclusions

In this article, new approximations to the inverse involute function have been presented. They are very accurate for a low degree polynomial, requiring a much reduced number of operations. In general, the balance between the computation efficiency and accuracy of the resulting rational approximations is very favorable for most engineering applications, including the following example, which is valid for $u \le 35^\circ$:

$$u = \mathrm{inv}^{-1}(x) \simeq \frac{x^{1/3}}{0.693357 + 0.192484 \cdot x^{2/3}}. \quad (30)$$

Although some specific approximations with a limited value of $u = 45^\circ$ and rational functions of [1, 2] and [2, 2] degrees have been presented here, we have also described the theoretical principles and the algorithms for applying them, which allow other approximations to result in larger or smaller degrees or on larger or smaller angle values, even in intervals that do not start from zero, which would generate much lower errors when reducing the approximation interval. This has been completed in this article with constraint imposition applied to the Remez algorithm. Additionally, it provides a rational approximation to the direct involute function as a new example of application, which avoids computation of the tangent function or the cube root in an eventual expansion in a series. Finally, the polar equation of the involute curve has been approximated with a very low error. It will allow a simplification of many other calculations in gear engineering. Some examples of its application have been presented.

Nomenclature

- U: Pressure angle in the involute curve. As we want to obtain approximations of u, we will consider it a function of the independent variable x; $u = \mathrm{inv}^{-1}(x)$
- u': Operating pressure angle used at point 6
- c: $c = \tan(u)$ is the intermediate variable used in (6)
- u_1, u_2, u_3: Intermediate values of the pressure angle u in the iterations of the Laskin formulation
- x: Polar angle, as the involute of the pressure angle u. We use it too as an independent variable working in the interval $[a, b]$
- t: Intermediate variable resulting of the transformation $t = x^{2/3}$ used in (4)
- y: Intermediate variable resulting of the transformation $y = x^{1/3}$ used at Section 3.3
- v: Variable resulting of transformation (9) working in the interval $[-1, 1]$
- a, b: Limits of the interval of values of x in (9)
- $T_i(x)$: Chebyshev polynomial of degree i for the variable x
- $J_i(x)$: Jacobi polynomial of degree i for the variable x
- a_n: Coefficients of a Chebyshev series
- r_n: Norm of the function a_n in (12)
- $w(v)$: Weight function in the calculation of the coefficients in orthogonal polynomial series
- $f(x)$: A general function in terms of the independent variable x
- δ, γ: Parameters in the weight function of Jacobi polynomials
- $E_{\max}$: Maximum error of an approximation
- z_i: Number of teeth in the gear i
- r: Pitch radius of a gear
- r': Functioning pitch radius of a gear
- s_i: Profile shift of the gear i.

Data Availability

The data used to support the findings of this study have been deposited in the library repository of the Universidad Francisco de Vitoria at http://ddfv.ufv.es/handle/10641/1435.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Supplementary Materials

The supplementary material is an Excel file including all the calculations of the different mathematical expressions of the inverse involute present in the paper. (*Supplementary Materials*)

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