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Analysis of selfsimilar solutions and a comparison principle for an heterogeneous diffusion cooperative system with advection and non-linear reaction

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Abstract

The present analysis introduces a system of cooperative species formulated with a high order parabolic operator, a Fisher-KPP reaction and a linear advection. Firstly, the oscillatory behaviour of solutions is shown to exist with a shooting method approach. It is to be highlighted that the existence of oscillatory patterns (also called instabilities) is an inherent property of high order operators. Afterwards, existence and uniqueness results are provided. The most remarkable result, obtained during the existence exercise, is related with the finding of a particular time-degenerate bound for the advection term that ensures positivity of solutions. This is one of the main results as such positivity property does not hold for high order operators in general. Indeed, high-order operators provide oscillatory solutions that may induce such solutions to be negative in the proximity of the null state introduced by the Fisher-KPP reaction term. As a consequence, a comparison principle does not hold as formulated in order two operators. Further, a positive maximal kernel with similar asymptotic behaviour compared to the high order kernel has been shown to exist and a precise assessment has been done with a computational exercise. Eventually, such a positive maximal kernel permits to show the existence of a comparison principle.

Keywords Fisher-KPP reaction · High-order high order · Oscillations · Positivity · Advection · Uniqueness

Mathematics Subject Classification 35K92 · 35K91 · 35K55

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1 Problem description and objectives

The theory of high order operators has attracted interest in the last years. One of the most challenging aspects is related to the loss of regularity and the impossibility to formulate a comparison principle. One of the approximations followed to further understand the high order operators implications has consisted on studying already known problems formulated with a classical order two operators. A remarkable study, by Galaktionov and Pohozaev (2002), proposed the analysis of the Fujita exponent to segregate between the existence of small global solutions and blow up. In the same analysis, the authors studied the existence of a majoring operator for which solutions order is preserved, and consequently maximum and comparison principles can be shown to exist. Note that the oscillatory properties introduced by the high order operators may induce a negative blow up, in the sense that solutions can explode to $-\infty$. It has been shown in a set of papers (Mizoguchi and Yanagida 1997) and (Mizoguchi and Yanagida 1998), that there exists a sequence of critical exponents defining the blow-up behaviour. Note that this is a new property introduced by high order operators that do not occur for the classical second-order diffusion.

The use of high order operators in the applied sciences comes together with the mathematical techniques development. Some important achievements are related to advances in De Giorgi's conjecture for an Allen-Cahn equation (Bonheure and Hamel 2017) and the Hardy inequalities (Galaktionov 2006).

This paper examines a system of cooperative species (f, g) with a Fisher-KPP type reaction and a time defined advection formulated with a high order operator. The problem will be referred as P:

$$\begin{aligned} f_t &= -\Delta^2 f + c(t)\cdot\nabla f + f(1-f) + g, \\ g_t &= -\Delta^2 g + c(t)\cdot\nabla g + g(1-g) + f, \\ \Delta^2 &= \frac{\partial^4}{\partial x^4}, \quad f_0(x);\, g_0(x) \in Y = L^\infty(\mathbb{R}) \cap L^1_{loc}(\mathbb{R}), \end{aligned} \tag{1}$$

where $c(t)$ represents a time-degenerate advection term and f_0, g_0 are the initial conditions in generalized Lebesgue spaces. Note that the reaction-absorption terms have been originally proposed by Kolmogorov et al. (1937) and Fisher (1937).

In addition, high order operators have been applied to characterize the oscillatory behaviour of solutions close the critical points (Rottschäfer and Doelman 1998), (Dee and Van Sarloos 1998). A formal analytical approach to the study instabilities induced by the high-order operators can be found in (Peletier and Troy 2001) and (Bonheure and Sánchez 2006).

Alternatively and along with the presented analysis, the qualitative assessments, a priori estimations and asymptotic approaches are validated with numerical simulations. Mainly, this article is concerned with the existence of oscillations, the characterization of existence and uniqueness, the finding of a time-degenerate advection function $c(\tau)$, $0 < \tau < t$ for positive solutions, the asymptotic behaviour of self-similar solutions and the existence of a majoring order-preserving operator for which a maximum principle can be formulated and whose asymptotic behaviour is similar to the oscillatory high order kernel.

2 General assessments

Firstly, some general assessments are introduced for the a single equation in P (1). This approach permits to introduce the properties expected to be exhibited by the high-order operator.

2.1 On the existence of oscillatory patterns

Consider the following problem for a single equation in P (1):

$$f_t = -\Delta^2 f + c(t) \cdot \nabla f + f(1-f), \quad f_0(x) = H(-x) \in Y. \tag{2}$$

Note that $H(-x)$ is a step function, particularly, the opposite Heaviside. This function is of relevance as it permits to study a positive mass evolution at $y < 0$ and null mass whenever $y > 0$ and asymptotically at $y \to \infty$. Along with the following lemma, and without loss of generality in the final conclusions, we will assume that the advection is constant, i.e. $c(t) = c$.

This alternative problem is intended to analyze the oscillatory patterns that close the equilibrium at $f = 0$ given a step-like initial data whose value is zero at ∞.

Lemma 2.1 *Oscillatory patterns exist as solutions to* (2) *for* $c < \frac{1}{6}$.

Proof Oscillatory patterns are shown for Navier boundary conditions given by $f(\infty, t) = \Delta f(\infty, t) = 0$. Under this approach, the following holds:

$$f'(\infty, t) = \theta, \quad f'''(\infty, t) = \vartheta, \tag{3}$$

where $\theta, \vartheta \in \mathbb{R}$ and f', f''' represent the first and third spatial derivatives respectively. Oscillatory patterns are shown to exist by a shooting method approach. For this purpose, the following Hamiltonian is defined (see (Bonheure and Hamel 2017) and (Peletier and Troy 2001) for a whole discussion on variational principles applied to high order operators):

$$J(f) = f''' f' - \frac{1}{2} f''^2 + c\, f + \frac{1}{3} f^3 - \frac{1}{2} f^2 + C, \tag{4}$$

where C is the integration constant obtained upon Hamiltonian first integration. Note that under the stationary solutions and in the asymptotic approach to them, the Hamiltonian satisfies:

$$\lim_{x \to \infty,\ x \to -\infty} J(f, f', f'', f''') = 0. \tag{5}$$

The integration constant, C, can be determined by considering the asymptotic condition $f = 1$. Such condition is obtained under the evolution of the positive mass in the step size initial condition $f_0(x) = H(-x)$ and in view of the reaction term stationary solutions:

$$J(f = 1) = c + \frac{1}{3} - \frac{1}{2} + C = 0, \tag{6}$$

to obtain, $C = \frac{1}{6} - c$. Any orbit preserves the Hamiltonian close the critical points and asymptotically ((Bonheure and Hamel 2017) and (Peletier and Troy 2001)). Consequently and considering the introduced Navier conditions:

$$J(f) = f'''(\infty) f'(\infty) + \frac{1}{6} - c = 0, \ \rightarrow \ \vartheta\, \theta + \frac{1}{6} - c = 0, \tag{7}$$

so that a relation between derivatives is given by:

$$\theta = \frac{c - \frac{1}{6}}{\vartheta}. \tag{8}$$

Note that for $c < \frac{1}{6}$ the third and first derivatives have changing sign whenever $\to \infty$ while for $c \geq \frac{1}{6}$ not. Aiming to show existence of oscillatory patterns, an spatial localizing variable is introduced (called γ) to analyze the asymptotic behaviour close the equilibrium at $f = 0$:

$$\gamma(\theta) = \sup\{x > 0, \quad f'(\theta, \vartheta(\theta), \cdot) < 0\}. \tag{9}$$

Now, admit:

$$\theta^* = \sup\{f' \ , \ with \ f < 1\}, \qquad \vartheta^* = \sup\{f''' \ , \ with \ f < 1\}. \tag{10}$$

The existence of a finite $\gamma(\theta)$ permits to show the oscillatory character of solutions when evolving to the stationary. To this end, admit $\theta = -\frac{1}{\gamma(\theta)}$. Considering the expression (8), a relation between ϑ and θ is obtained:

$$-\frac{1}{\gamma(\theta)}\vartheta = c - \frac{1}{6}. \tag{11}$$

Consider $\theta = \theta^*$ (where θ^* leads to the maximum of γ) and $\vartheta = \vartheta^*$, then the following holds for γ:

$$\gamma(\theta^*) = \frac{\vartheta^*}{\frac{1}{6} - c}. \tag{12}$$

Remind that $c < \frac{1}{6}$, which provides a positive value of the localizing variable γ. Note that an heteroclinic connection between critical points $f = 0$, $f = 1$ has a finite γ. Indeed, note that for the stationary solutions, the following holds:

$$\lim_{x \to +\infty}(f, f', f'', f''') = (0, 0, 0, 0), \quad \lim_{x \to -\infty}(f, f', f'', f''') = (1, 0, 0, 0), \tag{13}$$

so that, a non-trivial solution has a finite and maximum value of f''' named as ϑ^*. This permits to ensure that $\gamma(\theta^*)$ is finite.

Now, it is the intention to search for a finite $x > \gamma(\theta^*)$, so that the first derivative is positive in $(\gamma(\theta^*), x)$. For this purpose, admit the following definitions:

$$\varsigma(\theta) = \sup\{(x - \gamma(\theta^*)) > 0, \quad f'(\theta, \vartheta(\theta), \cdot) > 0, \ \forall x \in \ (\gamma(\theta^*), x)\}. \tag{14}$$

Now, define

$$\theta^{**} = \inf\{f', \quad f(\theta, \vartheta(\theta), \varsigma(\theta)) > 0\}, \quad \vartheta^{**} = \inf\{f''', \quad f(\theta, \vartheta(\theta), \varsigma(\theta)) > 0\}. \tag{15}$$

Admit a value for θ so that the solution is non-decreasing $(\gamma(\theta^*), x)$. Note that θ is positive, and in accordance to (8), ϑ is negative as $c < \frac{1}{6}$, then:

$$\theta = \frac{\sigma}{\varsigma - \gamma}, \tag{16}$$

where σ represents the step in $(\gamma(\theta^*), x)$. Considering (8), the following holds:

$$\frac{\sigma}{\varsigma - \gamma}\vartheta = c - \frac{1}{6}. \tag{17}$$

So that, ϑ is obtained as:

$$\varsigma = \gamma - \frac{\sigma \, \vartheta}{\left|c - \frac{1}{6}\right|}. \tag{18}$$

Admit now the minimum value of (ϑ^{**}), then the following expression provides the maximum spatial position:

$$\varsigma(\theta^{**}) = \gamma - \frac{\sigma \, \vartheta^{**}}{\left|c - \frac{1}{6}\right|}. \tag{19}$$

The assessments above have permitted to show that heteroclinic connections between the stationary $f = 0$ and $f = a$ are non-increasing in $(0, \gamma(\theta^*))$ and non-decreasing

$(\gamma(\theta^*), \varsigma(\theta^{**}))$, where $\gamma(\theta^*)$ and $\varsigma(\theta^{**})$ have been shown to be finite . This particular monotonic condition concludes on the existence of oscillatory patterns for $c < \frac{1}{6}$, while for $c \geq \frac{1}{6}$, the first and third derivatives have the same sign which is a typical behaviour of purely decreasing or increasing functions (non-oscillatory).

□

The following subsections introduce existence and uniqueness topics making use of analytical and numerical assessments

2.2 Existence and uniqueness for a single equation

The following norm is introduced on a generalized Sobolev space H^4_π:

$$\|f\|_\pi = \int_{\mathbb{R}} \pi(v) \sum_{n=0}^{4} \left|D^n f(v)\right| dv, \tag{20}$$

where $D = \frac{d}{dv}$ and

$$\pi(v) = e^{A_0|v|^{\frac{4}{3}} - \frac{1}{v}\frac{1}{t^\omega}\int_0^t (\|f_x(\tau)\|^2_{L^1_\pi})\, d\tau}, \tag{21}$$

with $\omega > 2$ (Montaru 2014) and $A_0 > 0$ being a small constant Galaktionov (2012). The introduced exponential weight admits the embedding $f \in H^4_\pi \subset L^1_\pi$ (see (Escobedo and Kavian 1987) for further dedicated analysis). In addition, the introduced space $f \in H^4_\pi \subset L^1_\pi$ with the associated norm $\|f\|_\pi$ is a Banach space, i.e. $\|f_1 + f_2\|_\pi \leq \|f_1\|_\pi + \|f_2\|_\pi$. This can be shown by standard techniques.

Before entering into the existence and uniqueness discussions, the solution to the associated homogeneous equation is characterized. To this end, admit only $f_t = -\Delta^2 f$. A particular kernel $F(x,t)$ can be obtained operating within selfsimilarity solutions:

$$F(x,\tau) = \tau^{-\frac{1}{4}}\varphi(\eta); \quad \eta = \frac{\|x\|}{\tau^{1/4}}, \tag{22}$$

for $0 < \tau \leq t$. Note that the profile φ is symmetric and radially distributed, so that upon substitution into the homogeneous equations, the following equation of elliptical type holds:

$$-\varphi^{(4)} + \frac{1}{4}\varphi'\eta + \frac{1}{4}\varphi = 0, \tag{23}$$

such that the normalization condition $\int_{\mathbb{R}} \varphi = 1$ holds and suggests that the mass of φ is null at infinity.

Theoretical analysis to (23) provided in Levin (1969) and Chavez and Galaktionov (2001) show the existence of oscillating patterns. Along the presented study, a solution to (23) is provided with a numerical exercise making use of Matlab software (in particular the bvp4c function). To this end, the scaled domain η has been splited into 10^5 nodes between the interval (0, 100). This interval has been considered sufficiently large to assess the asymptotic behaviour of φ at ∞. To start the Matlab collocation routine, it is required to introduce a first iterative function, in this case, given by a step like Heaviside condition belonging to the functional space Y introduced in (1). In addition, the following boundary type conditions have been considered to ensure the radially symmetric profile: $\varphi(0) = 1,\ \varphi'(0) = 0,\ \varphi^{(3)}(0) = 0,\ \varphi(\infty) = 0,\ \varphi^{(k)}(\infty) = 0;\ k = 1, 2...$

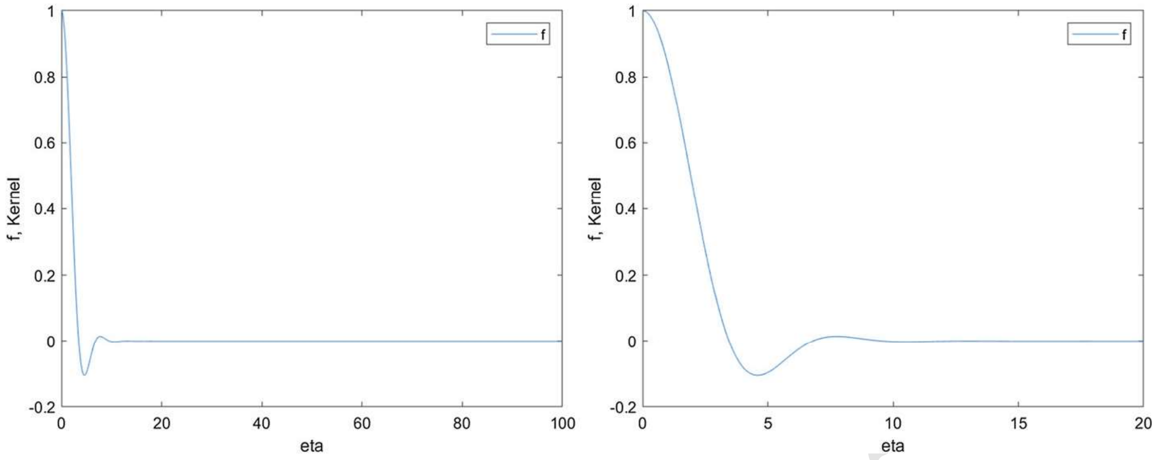

Fig. 1 Profile φ (named f in the figures) as solution to equation (23) in the interval [0, 100] (left) and zoom in the interval [0, 20]. Note the existence of oscillations that hold in the complete domain. The variable η is given horizontally

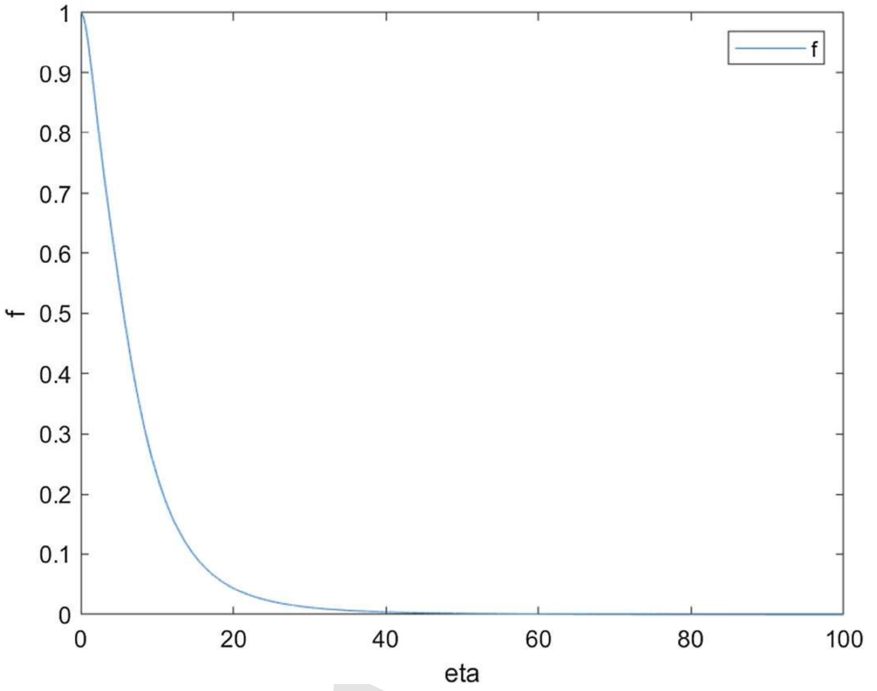

Fig. 2 Solution to (24) for $c = 1$ and $\tau = 0.1$. Note that at the beginning, $\tau << 1$, no oscillations occur

The Fig. 1 provides the profile obtained for the interval [0, 100] and a zoom in the interval [0, 20]. Note that the obtained profile associated with the high order homogeneous operator has oscillations and those oscillations are kept for any value of η just to change the scale and the interval of interest.

Now, the intention is to determine dedicated solutions for the whole equation (2) in the selfsimilar form as introduced in (22). After operating, the following elliptic equation holds:

$$-\varphi^{(4)} + \frac{1}{4}\varphi'\eta + \frac{1}{4}\varphi = c\,\tau^{-\frac{1}{2}}\varphi' + \varphi(1-\varphi), \tag{24}$$

with the initial condition given by a Heaviside type that satisfies:

$$\begin{gathered}\varphi(0) = 1,\ \ \varphi'(0) = 0,\ \ \varphi^{(3)}(0) = 0,\ \ \varphi(\eta \to \infty) = 0,\\ \varphi^{(k)}(\eta \to \infty) = 0,\ \ k = 1, 2...\end{gathered} \tag{25}$$

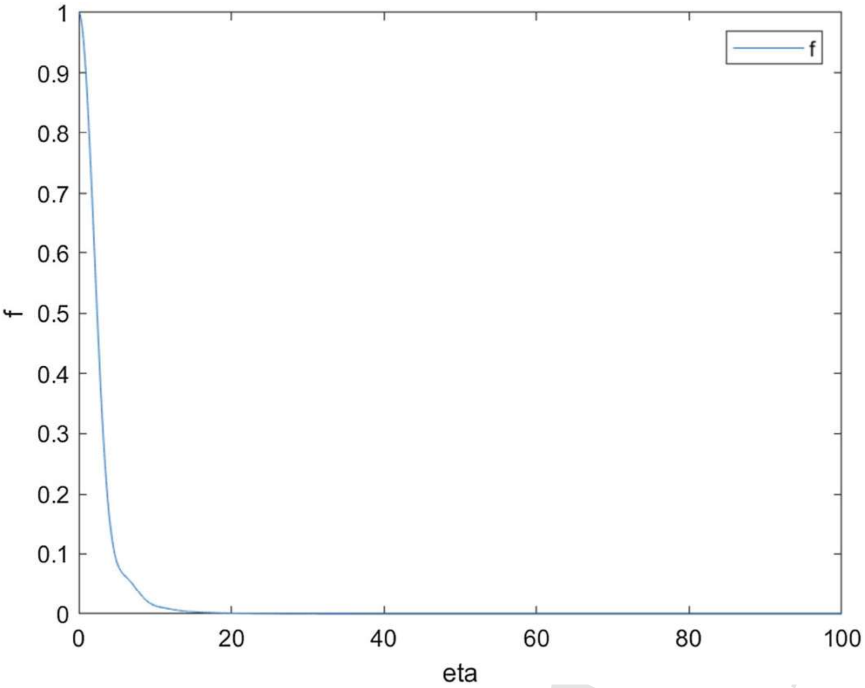

Fig. 3 Solution to (24) for $c = 1$ and $\tau = 1$. Still solutions are monotone with no oscillations

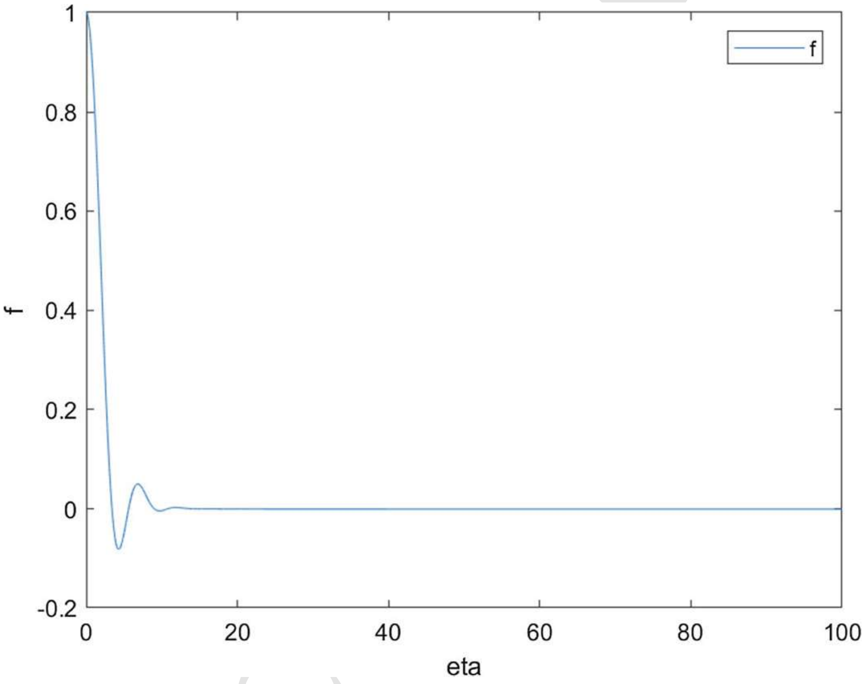

Fig. 4 Solution to (24) for $c = 1$ and $\tau = 5$. At some $1 < \tau < 5$ oscillations start to occur

Once the convection term is introduced (through the term $c\,\tau^{-\frac{1}{2}}\varphi'$), the profile behaves monotonically (no oscillation) for $\tau << 1$. Upon evolution the convective term is not predominant given the scale $\tau^{-\frac{1}{2}}$ compared to the oscillating kernel (Fig. 2). This is compiled in Figs. from 3, 4, 5, 6 and 7. In addition, the mentioned figures provide evidences of existence of profiles to (24) and, consequently to the problem P (2). 1

Note that Figs. 3, 4, 5 and 7 provide information on the amplitude of oscillations, increasing for increasing values of τ. In addition, if the balancing between the convection term and the

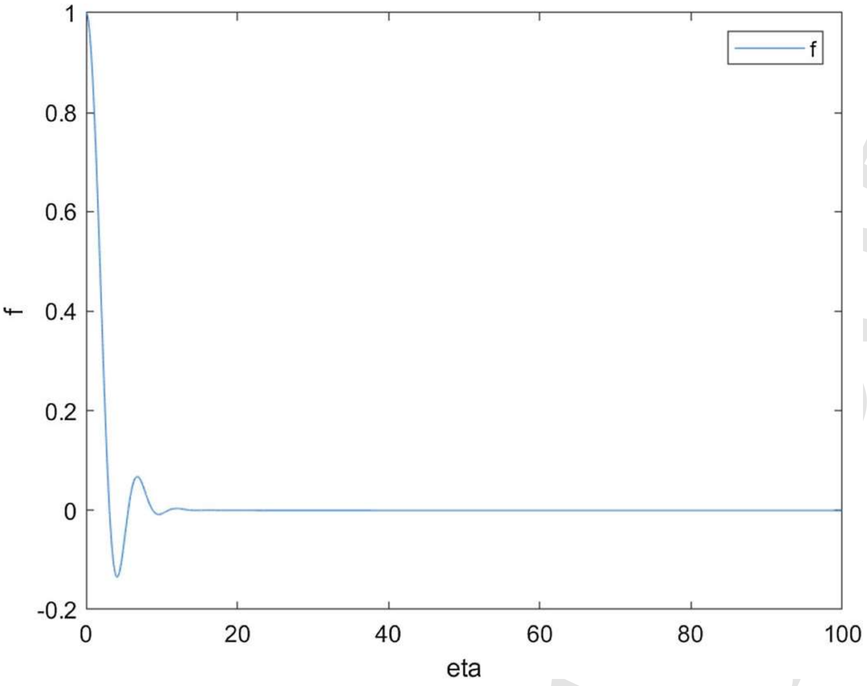

Fig. 5 Solution to (24) for $c = 1$ and $\tau = 10$

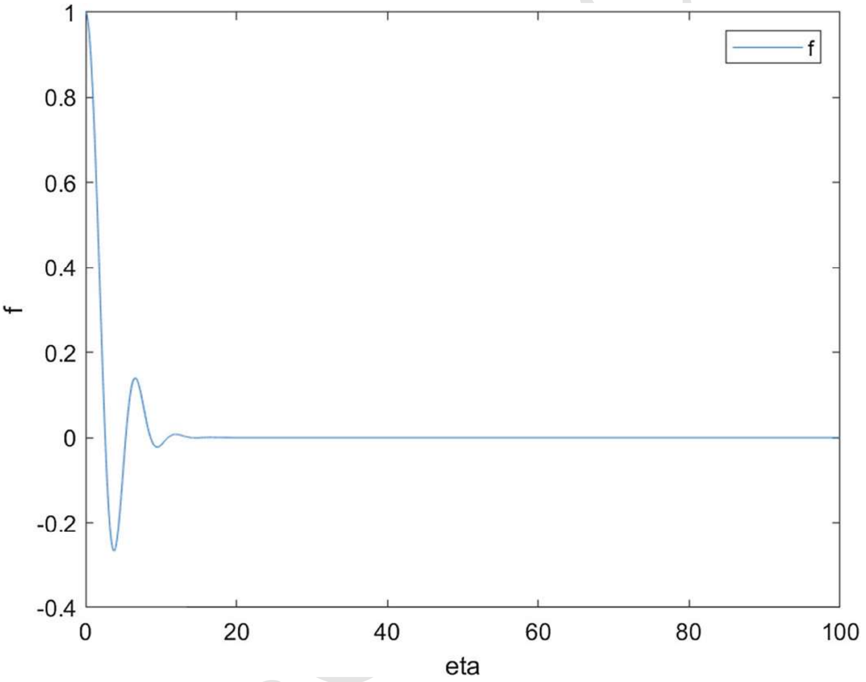

Fig. 6 Solution to (24) for $c = 1$ and $\tau = 1000$

time variable is kept such that $c\,\tau^{-\frac{1}{2}} = K_c \sim 1$, solutions do not oscillate. A sharp assessment for K_c is provided with numerical explorations associated to the analytical solutions in the selfsimilar form. The results are provided in Figs. 8, 9 and 10, so that for

$$c\,\tau^{-\frac{1}{2}} \leq K_c = 2.112, \tag{26}$$

profiles do not oscillate. The value for K_c has been obtained with three decimal numbers and with a contact of order 10^{-4} between the profile and the horizontal axis (see Fig. 9). Note that the contact might be further precise $< 10^{-4}$, nonetheless this does not impact the

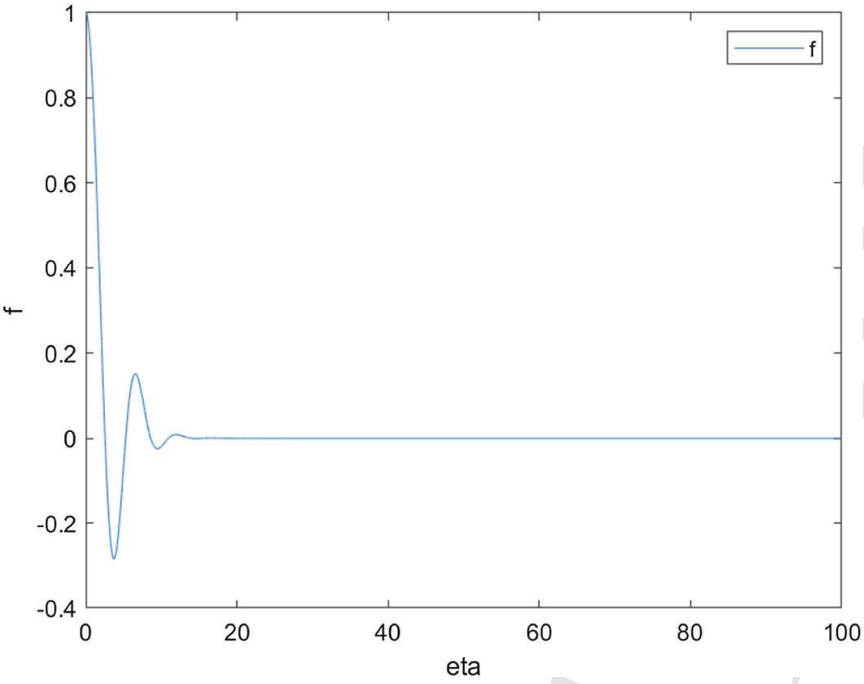

Fig. 7 Solution to (24) for $c = 1$ and $\tau \to \infty$

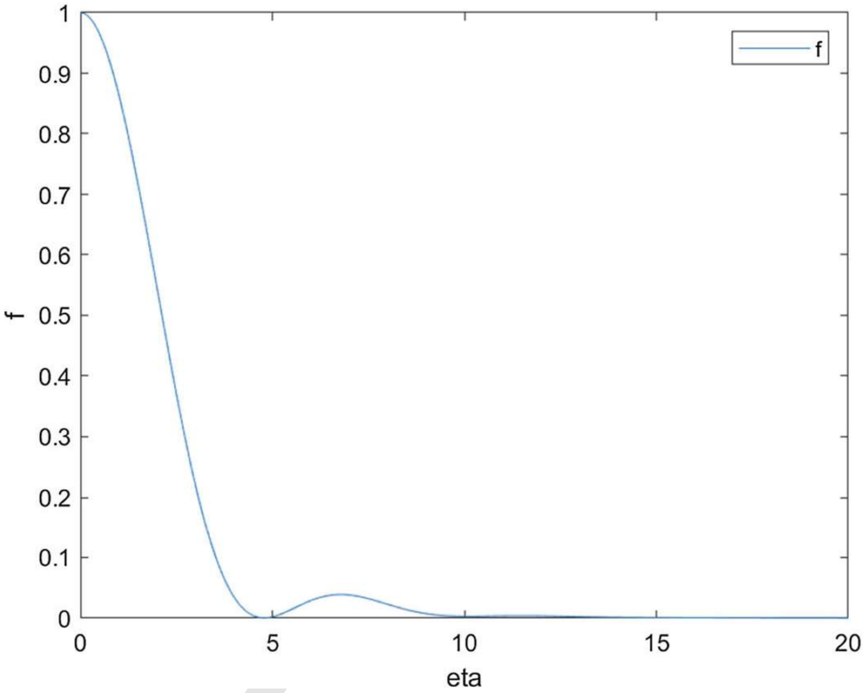

Fig. 8 For $c\,\tau^{-\frac{1}{2}} = K_c = 2.112$, profiles to (24) are positive

third decimal position in K_c. Consequently, it is considered that the assessment on K_c is sufficiently precise.

Note that if the convection is considered as a temporal evolution, it is possible to obtain a maximal degenerate with time behaviour (not defined in the proximity of τ null) of the form:

$$c(\tau) \leq 2.112\,\tau^{\frac{1}{2}}, \quad 0 < \tau < t \leq \infty, \tag{27}$$

such that solutions are positive.

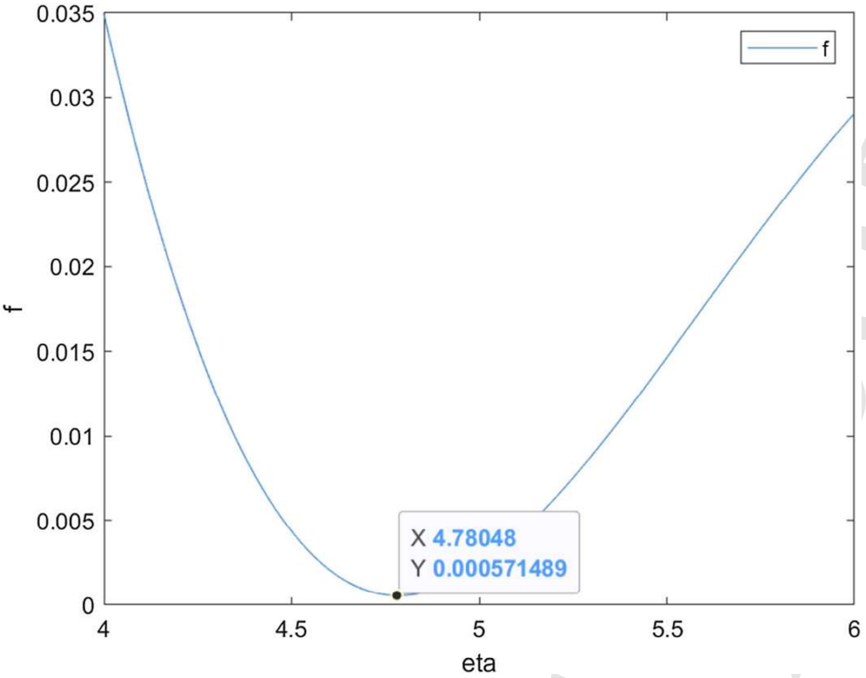

Fig. 9 Zoom of previous Fig. 8 to show that solutions are indeed kept positive

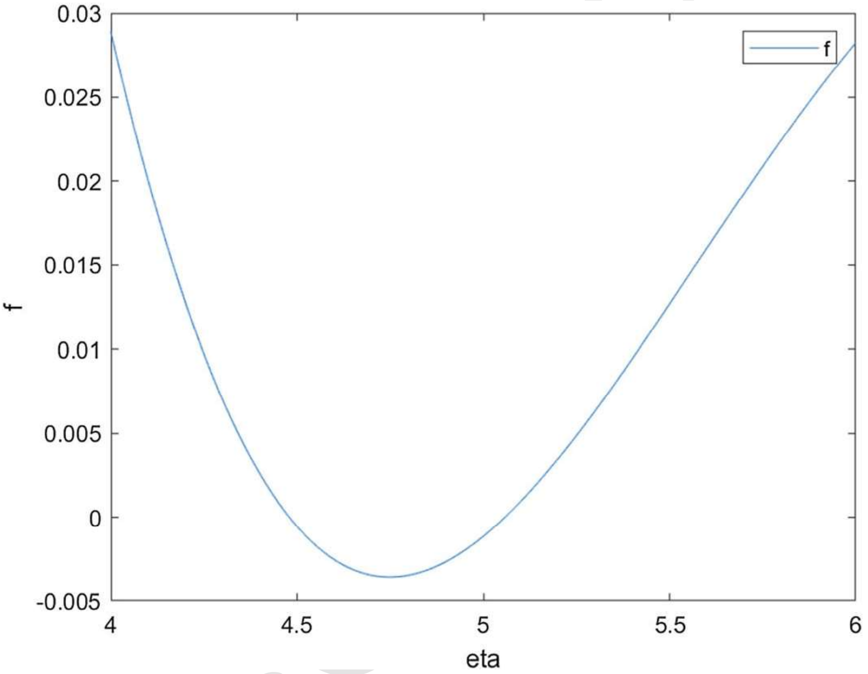

Fig. 10 For $c\,\tau^{-\frac{1}{2}} = 2.2$, profiles to (24) start to be negative

In addition, if the advection term in (24) is time predominant compared with the $\tau^{-\frac{1}{2}}$ degenerate function, solutions are positive for $\tau >> 1$. The most simple one can be considered as:

$$c(\tau) \sim \tau, \tag{28}$$

so that the advection term behaves $\sim \tau^{\frac{1}{2}}$ in (24). Note that this particular behaviour in the advection is validated with numerical results for the complete system.

The next lines sought to analyze uniqueness considering the generalized Sobolev space H^4_π together with the norm (20). To this end the map $E : H^4_\pi \to H^4_\pi$ given by

$$\begin{aligned} E(f) &= F(x,\tau) * f_0 \\ &\quad + \int_0^\tau \{ F(x,\tau - r) * (c \cdot \nabla f(x,r)) + F(x,\tau - r) * f(x,r)(1 - f(x,r))\} dr, \end{aligned} \tag{29}$$

where c has been considered as constant without loss of generality and in the sake of simplicity. The operator E (29) has a single fix point $f(x,\tau) = E(f(x,\tau))$. For this purpose:

$$\begin{aligned} &\|E(f_1) - E(f_2)\|_\pi \\ &\quad \le \int_0^\tau \{\|c \cdot \nabla F(x,\tau - r) * (f_1 - f_2)\|_\pi + \|F(x,\tau - r) * [f_1(1 - f_1) - f_2(1 - f_2)]\|_\pi\} dr \\ &\le N \int_0^\tau \int_\tau^r \{c\,\|(f_1 - f_2)\|_\pi + \|[f_1(1 - f_1) - f_2(1 - f_2)]\|_\pi\} dw\, dr, \end{aligned} \tag{30}$$

where $N = \sup\{\sup\{\|F(x,\tau - r - w)\|_\pi \ , \|\nabla F(x,\tau - r - w)\|_\pi\}, \ \forall x \in \mathbb{R}, \ \forall \tau > 0\}$ and for any $r > 0$ and $w > 0$. Note that the integrand is contractive. The non-linear terms can be further assessed:

$$\begin{aligned} \|[f_1(1 - 1_1) - f_2(1 - f_2)]\|_\pi &= \int_{\mathbb{R}} \pi(v) \sum_{n=0}^{4} \left| D^n [f_1(1 - f_1) - f_2(1 - f_2)] \right| dv \\ &= \int_{\mathbb{R}} \pi(v) \left\{ |f_1(1 - f_1) - f_2(1 - f_2)| + \sum_{n=1}^{4} \left| D^n [f_1(1 - f_1) - f_2(1 - f_2)] \right| \right\} dv \\ &\le B \int_{\mathbb{R}} \pi(v) \left\{ |(f_1 - f_2)| + \sum_{n=1}^{4} \sum_{m=1}^{n} \left| \binom{n}{m} (f_1 - f_2)^{(m)} \right| \right\} dv \\ &= B^2 \int_{\mathbb{R}} \pi(v) \sum_{n=0}^{4} \left| D^n [f_1 - f_2] \right| dv = B^2 \, \|f_1 - f_2\|_\pi \ , \end{aligned} \tag{31}$$

such that $f_1, f_2 \in H^4_\pi$ and $B = \max\{1 - (f_1 - f_2), [f - (f_1 - f_2)]^{n-m}\}$. Therefore:

$$\|E(f_1) - E(f_2)\|_\pi \le N\,B \int_0^\tau \int_\tau^r \|f_1 - f_2\|_\pi \, dw\, ds = N\,B^2 \tau(\tau - r)\, \|f_1 - f_2\|_\pi \, . \tag{32}$$

Note that for any $0 < \tau < t < \infty$ and for any finite $r > 0$, E is a contractive map. Finally, uniqueness of solutions holds for $f_1 \swarrow f_2$ so that $E(f_1) \swarrow f_1$ in H^4_π.

2.3 Existence and uniqueness for the whole system

Following the philosophy exposed in Sect. 2.2, the intention is now to study the complete system P (1). To this end, numerical and analytical techniques are provided.

The set of equation in P (1) are represented making use of a symbolic matrix notation:

$$\begin{pmatrix} f \\ g \end{pmatrix}_t = \begin{pmatrix} -\frac{\partial^4}{\partial x^4} & 0 \\ 0 & -\frac{\partial^4}{\partial x^4} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} + c \cdot \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} f(1 - f) + g \\ g(1 - g) + f \end{pmatrix}, \tag{33}$$

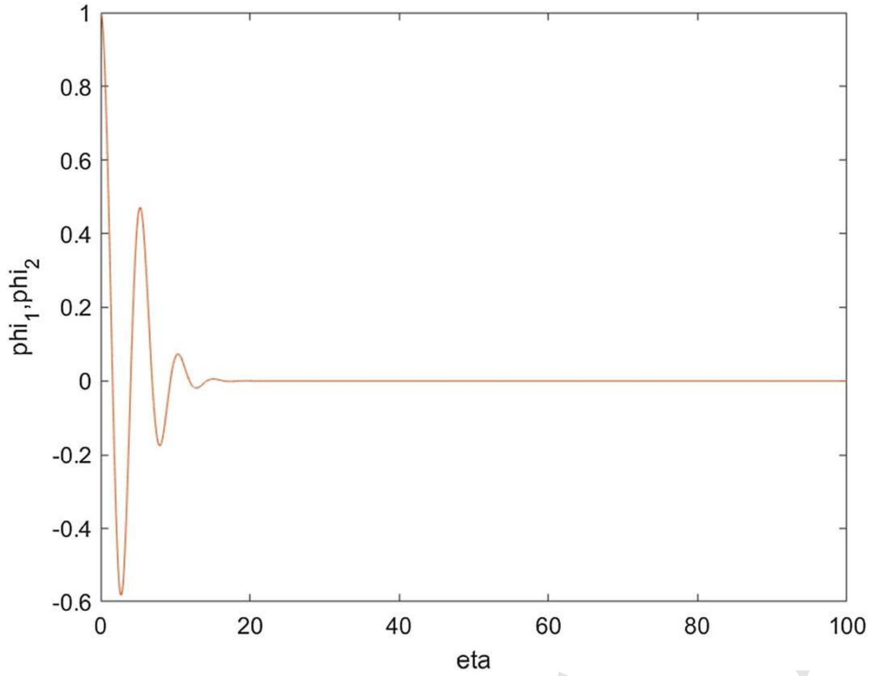

Fig. 11 Solutions φ_1 and φ_2 to (37). Given the exhibited symmetry between equations, both solutions are superimposed

$$f_0(x),\quad g_0(x) \in Y. \tag{34}$$

Admit that $H = \begin{pmatrix} f \\ g \end{pmatrix}$, then:

$$H_t = L_4\,H + c\cdot L_1\,H + Q(H), \tag{35}$$

where L_4 represents the fourth order symbolic matrix, L_1 the first order and $Q(H)$ the independent terms.

The solution to the associated homogeneous operator $H_t = L_4 H$ is first characterized with numerical means. To thin end, define

$$F_1(x,\tau) = \tau^{-\frac{1}{4}}\varphi_1(\eta),\quad F_2(x,\tau) = \tau^{-\frac{1}{4}}\varphi_2(\eta),\qquad \eta = \frac{\|x\|}{\tau^{1/4}}, \tag{36}$$

for $0 < \tau \leq t$. Note that both profiles φ_1 and φ_2 are symmetric and radially distributed. Upon replacement in the homogeneous problem, the following elliptic equations hold:

$$-\varphi_1^{(4)} + \frac{1}{4}\,\varphi_1'\eta + \frac{1}{4}\varphi_1 = 0,\quad -\varphi_2^{(4)} + \frac{1}{4}\,\varphi_2'\eta + \frac{1}{4}\varphi_2 = 0, \tag{37}$$

with the normalization $\int_{\mathbb{R}}\varphi_1 = 1$ and $\int_{\mathbb{R}}\varphi_2 = 1$ and subjected to the boundary conditions:

$$\varphi_1(0) = 1,\ \varphi_1'(0) = 0,\ \varphi_1'''(0) = 0,\ \varphi_1(\infty) = 0,\ \varphi_1^{(k)}(\infty) = 0,\ k = 1, 2..., \tag{38}$$

$$\varphi_2(0) = 1,\ \varphi_2'(0) = 0,\ \varphi_2'''(0) = 0,\ \varphi_2(\infty) = 0,\ \varphi_2^{(k)}(\infty) = 0,\ k = 1, 2... \tag{39}$$

Note that solutions φ_1 and φ_2 are provided in Fig. 11.

Now, an following the same idea as exposed in the previous section, the intention is to determine dedicated solutions in the selfsimilar profiles to equation (33). The system reads:

$$-\varphi_1^{(4)} + \frac{1}{4}\,\varphi_1'\,\eta + \frac{1}{4}\varphi_1 = c\,\tau^{-\frac{1}{2}}\varphi_1' + \varphi_1(1-\varphi_1) + \varphi_2,$$

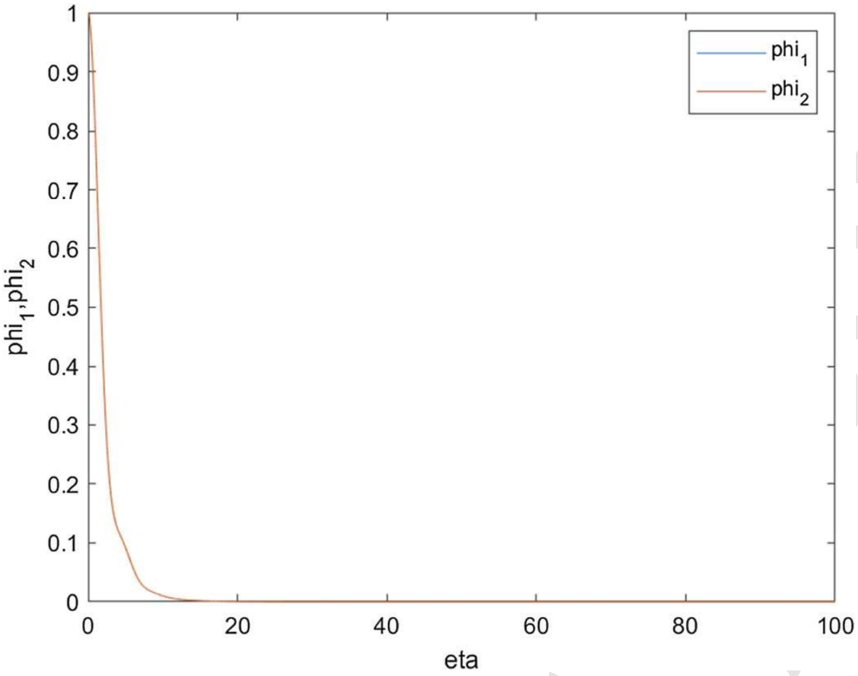

Fig. 12 Solutions to (40) for $c = 1; \ \tau = 0.1$. Note that solutions are superimposed and positive

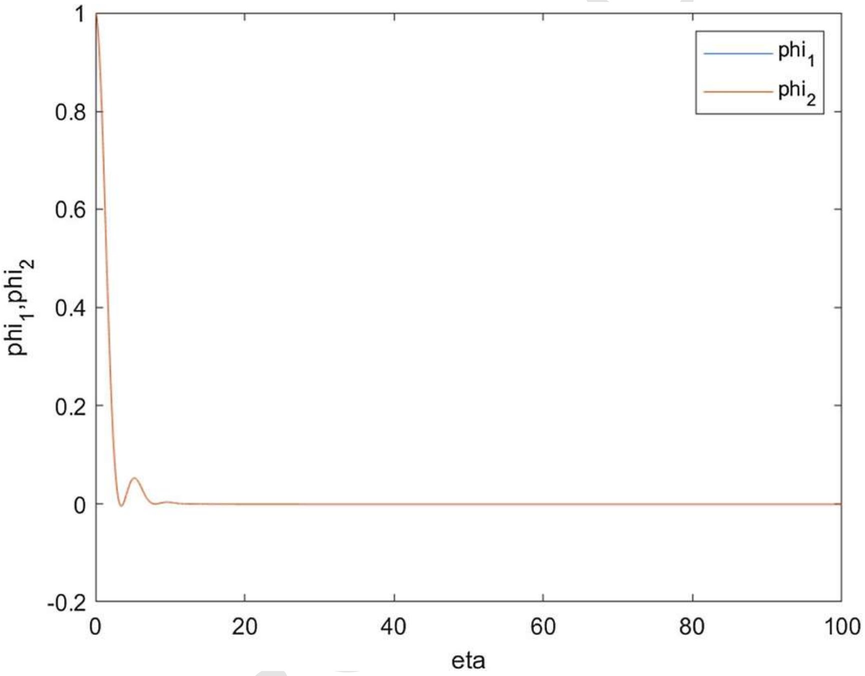

Fig. 13 Solutions to (40) for $c = 1; \ \tau = 0.212$. Note that solutions are superimposed and positive

$$-\varphi_2^{(4)} + \frac{1}{4}\varphi_2'\eta + \frac{1}{4}\varphi_2 = c\,\tau^{-\frac{1}{2}}\varphi_2' + \varphi_2(1-\varphi_2) + \varphi_1, \tag{40}$$

with the set of boundary conditions given in (38) and (39).

Solutions to (40) for $c(\tau) \sim \tau; \ \tau = 1$. Note that solutions are superimposed and oscillatory

Existence of solutions to (40) are given based on a numerical computation with Matlab and for different values in the evolution parameter τ, see Figs. 12, 13, 14 and 15. Note that

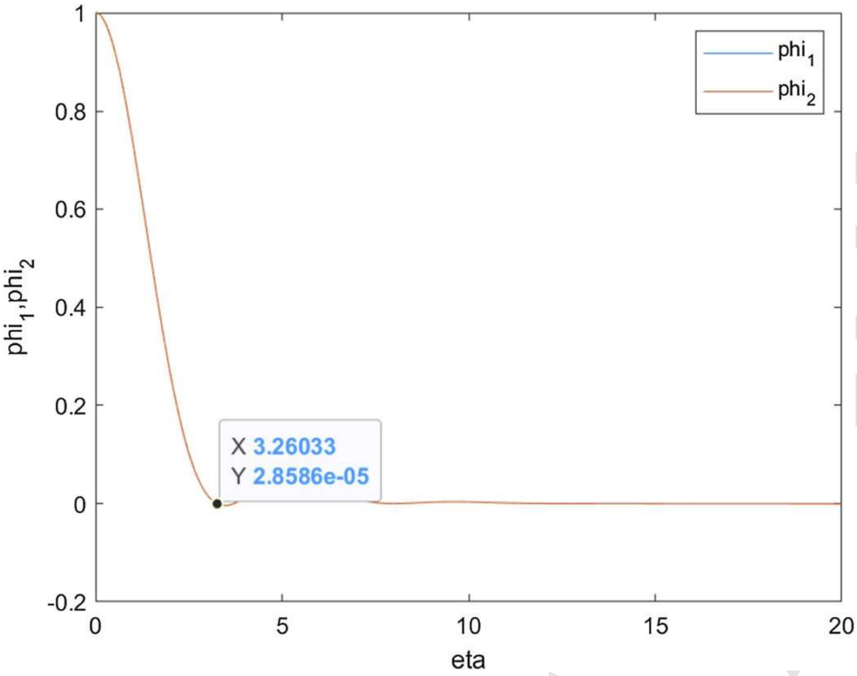

Fig. 14 Solutions to (40) for $c = 1;\ \ \tau = 0.2012$ with a zoom on the interval [0, 20]. Note that solutions are superimposed and positive

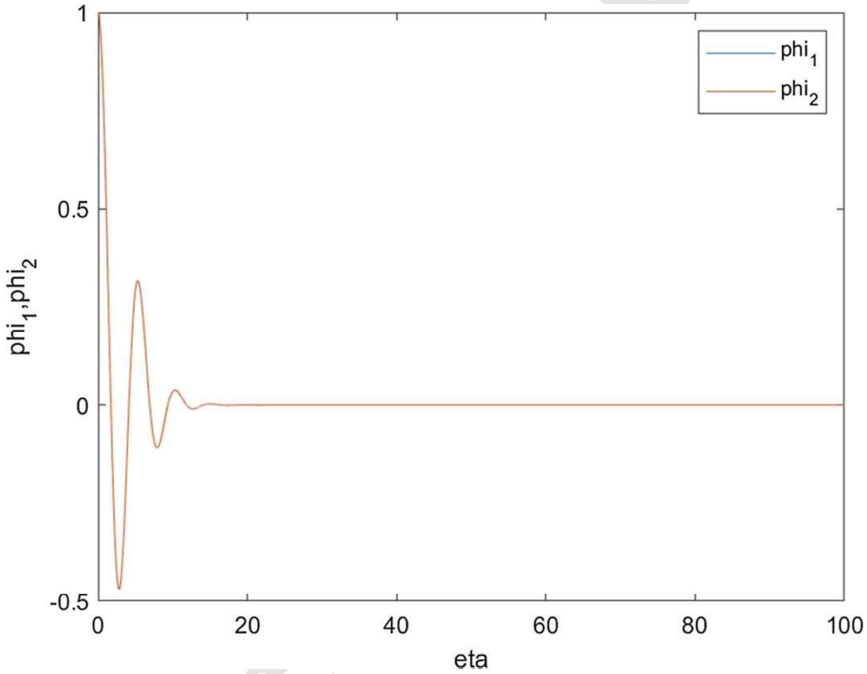

Fig. 15 Solutions to (40) for $c = 1;\ \ \tau = 10$. Note that solutions are superimposed and oscillatory

solutions are kept positive provided the following relation holds in accordance with Fig. 14:

$$c\,\tau^{-\frac{1}{2}} \leq K_s = 1\ \cdot\ 0.212^{-\frac{1}{2}} = 2.171. \tag{41}$$

The value for K_s has been obtained with three decimal numbers and with a contact of order 10^{-4} between the profile and the horizontal axis. Such contact can be reduced (see Fig. 14, for a contact $< 10^{-4}$), nonetheless this does not impact the obtained value of K_s up to the third decimal.

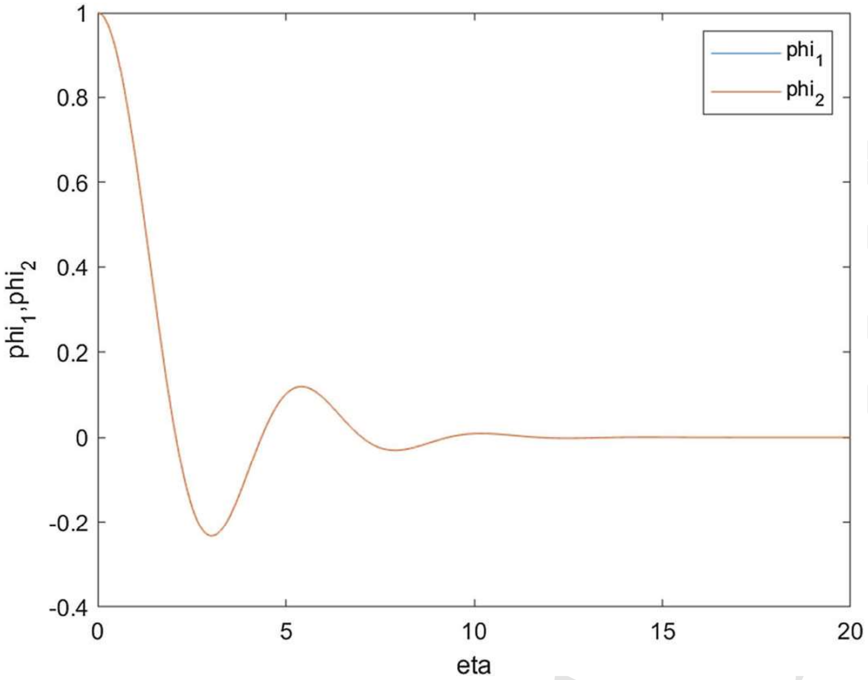

Fig. 16 Solutions to (40) for $c(\tau) \sim \tau;\ \tau = 4.809$. Note that solutions are superimposed and start to be positive

Then, a time degenerate convection satisfying:

$$c(\tau) \leq 2.171\tau^{\frac{1}{2}}, \tag{42}$$

preserves the positivity condition along the profile evolution. As disccused in the previous section a time defined advection of the form $c(\tau) \sim \tau$ leads to positive solutions for $\tau >> 1$. The validation of this last statement is provided in Figs. 16, 17 and 18.

$$c(\tau) \sim \tau, \tag{43}$$

To show uniqueness, admit the following vectors of solutions:

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{44}$$

Consider the following defined norm:

$$\|F\|_\pi = \int_{\mathbb{R}} \pi(v) \sum_{k=0}^{3} \left| D^k(\nabla \cdot F(v)) \right|^2 dv + \int_{\mathbb{R}} \pi(v) \sum_{i=1}^{2} \sum_{j=1}^{2} f_i(v) g_j(v) dv, \tag{45}$$

which can be proved to generate a Banach space making use of standard means.

Note that the divergence is considered in the following maximal sense:

$$\nabla \cdot F(v) = \max\{\frac{\partial f_1}{\partial v}, \frac{\partial f_2}{\partial v}\}. \tag{46}$$

To show uniqueness, it is required to define an appropriate map given by:

$$E : H_\pi^4 \times H_\pi^4 \to H_\pi^4 \times H_\pi^4, \tag{47}$$

such that:

$$E(F) = E(f_1, f_2) = M(x,t) * F_0(x)$$

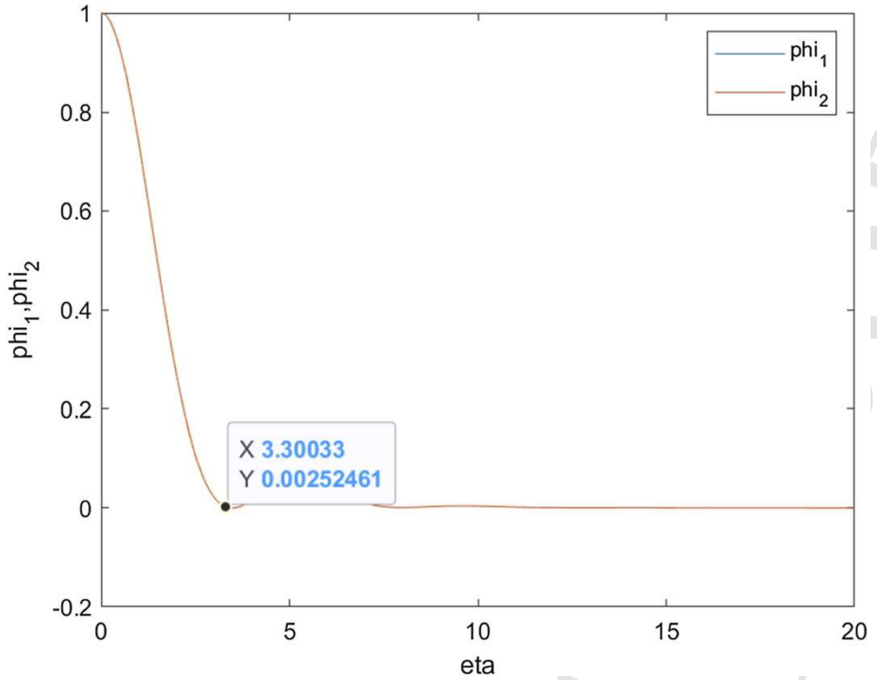

Fig. 17 Solutions to (40) for $c(\tau) \sim \tau;\ \tau = 20$. Note that solutions are superimposed and positive. This positivity condition is kept for larger values in τ

$$+\int_0^{\tau} [c \cdot \nabla M(x,\tau - r) * F(x,r) + M(x,\tau - r) * Q(F(x,r))]\,dr, \quad (48)$$

where

$$M = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad \nabla M = \begin{pmatrix} \nabla F_1 & 0 \\ 0 & \nabla F_2 \end{pmatrix} \quad (49)$$

F_1 and F_2 as defined in (36) and F_0 is a vector whose components are the initial data $f_0(x)$. It is required to show that the map E has a unique fixed point: $F(x,t) = E(F(x,t)) = E(f_1(x,t), f_2(x,t)) = (f_1(x,t), f_2(x,t))$,

$$\begin{aligned} \|E(F) - E(G)\|_{\pi} &\leq \int_0^{\tau} [c \cdot \nabla M(x,\tau - r) * (F - G)(x,r) \\ &\quad + M(x,\tau - r) * (Q(F(x,r)) - Q(G(x,r)))]\,dr \\ &\leq \Omega \int_0^{\tau} \int_{\tau}^{r} [\|F - G\|_{\pi}\,(\tau - r - p) + \|Q(F) - Q(G)\|_{\pi}\,(\tau - r - p)]\,dp\,dr, \end{aligned} \quad (50)$$

where $\omega = \sup\{\sup\{\|M\|_{\pi}\,, c \cdot \|\nabla M\|_{\pi}\},\ \ \forall x,\ \ 0 < p < r < \tau\}$. Note that:

$$\begin{aligned} \|Q(F) - Q(G)\|_{\pi} &= \int_{\mathbb{R}} \pi(v) \sum_{k=0}^{3} \left| D^k (\nabla \cdot (Q(F) - Q(G)) \right|^2 dv \\ &\quad + \int_{\mathbb{R}} \pi(v) \sum_{k=1}^{2} \sum_{i=1}^{2} (Q(F) - Q(G))_i (Q(F) - Q(G))_j dv. \end{aligned} \quad (51)$$

where,

$$Q(F) = \begin{pmatrix} f_1(1 - f_1) + f_2 \\ f_2(1 - f_2) + f_1 \end{pmatrix}, \quad Q(G) = \begin{pmatrix} g_1(1 - g_1) + g_2 \\ g_2(1 - g_2) + g_1 \end{pmatrix}, \quad (52)$$

then,

$$Q(F)-Q(G)=\begin{pmatrix} f_1-g_1+f_2-g_2+(g_1+f_1)(g_1-f_1)\\ f_2-g_2+f_1-g_1+(g_2+f_2)(g_2-f_2)\end{pmatrix}$$
$$=\begin{pmatrix}(1-(f_1+g_1))(f_1-g_1)\\(1-(f_2+g_2))(f_2-g_2)\end{pmatrix}+\begin{pmatrix}f_2-g_2\\f_1-g_1\end{pmatrix}\leq\begin{pmatrix}\omega_1(f_1-g_1)\\\omega_1(f_2-g_2)\end{pmatrix}+\begin{pmatrix}f_2-g_2\\f_1-g_1\end{pmatrix}$$
$$\sim\begin{pmatrix}\omega_1(f_1-g_1)\\\omega_2(f_2-g_2)\end{pmatrix}+\begin{pmatrix}f_1-g_1\\f_2-g_2\end{pmatrix}. \tag{53}$$

Now, admit that $f_1>g_1$ and $f_2>g_2$, then:

$$\begin{pmatrix}(\omega_1+1)(f_1-g_1)\\(\omega_2+1)(f_2-g_2)\end{pmatrix}\leq\Omega_1(F-G). \tag{54}$$

where $\omega_1=|1-(f_1+g_1)|$ and $\omega_2=|1-(f_2+g_2)|$ and

$$\Omega_1=\max\{\omega_1+1,\omega_2+1\}. \tag{55}$$

Coming to (51)

$$\|Q(F)-Q(G)\|_\pi\leq\int_{\mathbb{R}}\pi(v)\sum_{k=0}^{3}\left|D^k(\nabla\cdot\Omega_1(F-G)\right|^2dv$$
$$+\int_{\mathbb{R}}\pi(v)\sum_{i=1}^{2}\sum_{j=1}^{2}\Omega_1(F-G)_i\,\Omega_1(F-G)_j dv=\Omega_1^2\,\|F-G\|_\pi\,. \tag{56}$$

After compilation, the following holds:

$$\|E(F)-E(G)\|_\pi\leq\Omega\int_0^\tau\int_\tau^r\left[\|F-G\|_\pi(\tau-r-p)+\Omega_1^2\,\|F-G\|_\pi(\tau-r-p)\right]dp\,dr,$$
$$=\Omega(1+\Omega_1^2)\tau(\tau-r)\,\|F-G\|_\pi\,. \tag{57}$$

Then, for finite values of τ and r. i.e. for a local evolution, uniqueness holds. Indeed, for $G\swarrow F$, the operator is such that $E(G)\swarrow E(F)$ in H^4_π according to the bound properties exhibit in (57). Consequently, there exists a unique fix point $E(F)=F$ locally defined.

3 Maximal monotone bounds and order preserving

In the proximity of the null stationary solution $f\sim 0$, $g\sim 0$, the problem P (1) admits a linearization in the independent terms. In addition and along the present section, define $S=f+g$, so that:

$$S_t=-\Delta^2 S+c\cdot\nabla S+2\ |S|,\quad S_0(x)=f_0(x)+g_0(x)\in Y \tag{58}$$

Consider the homogeneous equation only given as:

$$S_t=-\Delta^2 S,\ \ S_0(x)=H(-x)\ \in\ Y, \tag{59}$$

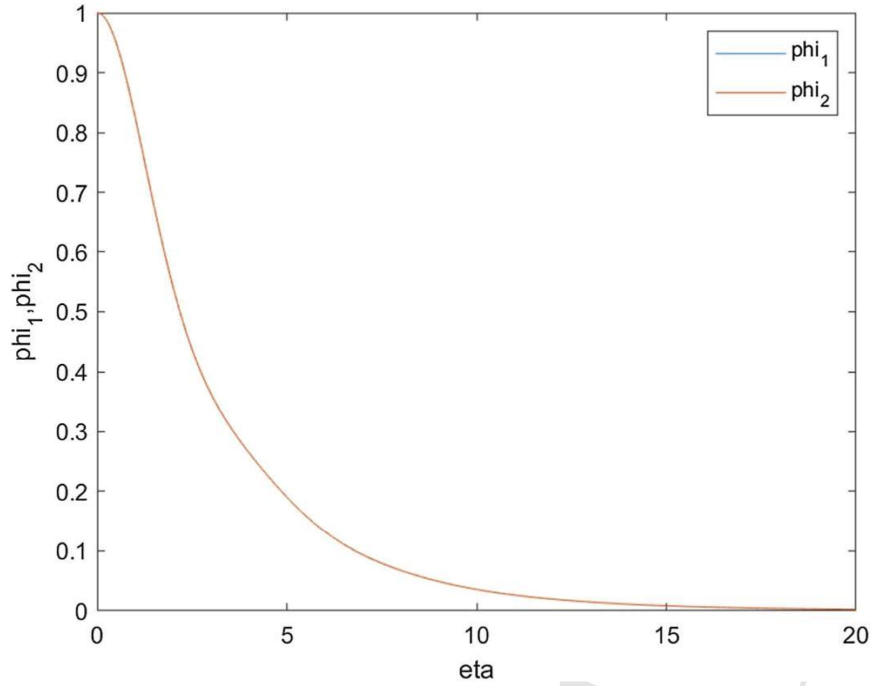

Fig. 18 Solutions to (40) for $c(\tau) \sim \tau; \tau = 20$. Note that solutions are superimposed and positive. This positivity condition is kept for larger values in τ

and admit the scaling:

$$p(x,t) = \tau^{-\frac{1}{4}} e^{\tau} \sigma(\eta), \quad \eta = \frac{\|x\|}{\tau^{-\frac{1}{4}}}. \tag{60}$$

Now, replace (60) into (59), so that the following elliptic equation holds:

$$-\sigma^4 + \frac{1}{4}\eta\sigma' + \frac{1}{4}\sigma = 0, \quad \int_{\mathbb{R}} \sigma = 1. \tag{61}$$

A bound for $\sigma(\eta)$ has been shown in Galaktionov (2001):

$$|\sigma(\eta)| \leq C_0\, \Sigma(\eta), \tag{62}$$

where $\Sigma(\eta) = \delta_1 e^{-c_0|\eta|^{\frac{4}{3}}} > 0, \ \ \delta_1 = \left(\int_{\mathbb{R}} e^{-c_0|\eta|^{\frac{4}{3}}} \right)^{-1}$ (Fig. 18).

The constant $C_0 > 0$ is referred as the order deficiency and shall be considered large enough so as to keep the maximality properties of Σ compared to σ (see Fig. 19 for particular values of c_0 and C_0 and reference (Galaktionov 2012) for further details).

Once suitable values of c_0 and C_0 have been explored and compiled in Fig. 19, it is possible to conclude on two different functions acting as kernels:

$$p(x,t) = \tau^{-\frac{1}{4}} e^{\tau} \sigma(\eta), \quad P(x,t) = \tau^{-\frac{1}{4}} e^{\tau} \Sigma(\eta), \tag{63}$$

The purely monotone kernel $P(x,t)$ has the positivity condition and behaves asymptotically as the oscillatory kernel $p(x,t)$. As a consequence, the following lemma to compare solutions holds:

Lemma 3.1 *Consider the majoring initial data,* $\tilde{S}_0(x)$, *such that* $\tilde{S}_0(x) \geq S_0(x)$, *then the order between solutions is preserved* $\tilde{S}(x,t) \geq S(x,t)$.

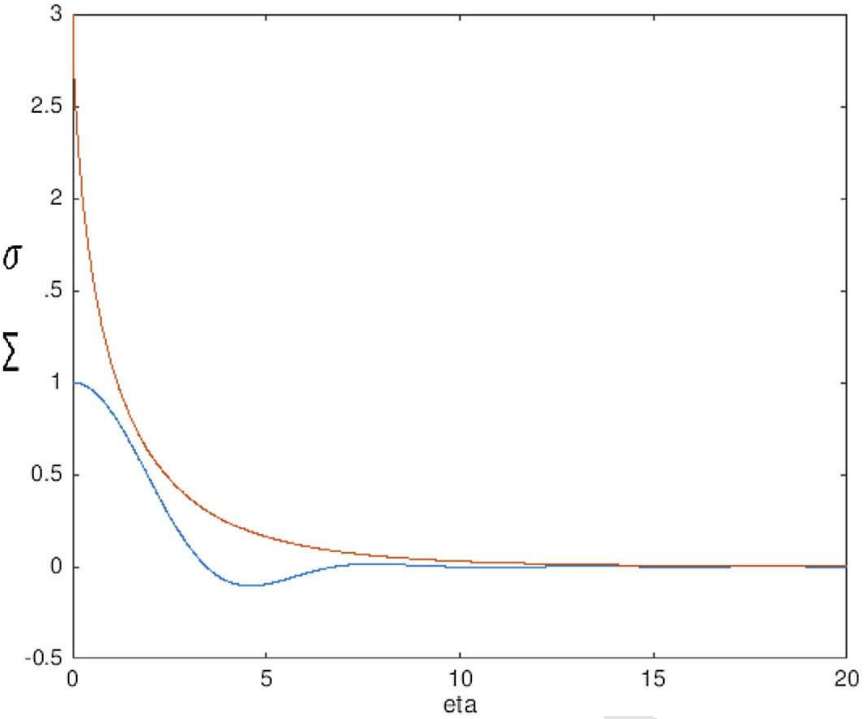

Fig. 19 Comparison between the oscillatory Kernel σ (blue line) and the monotonic majoring kernel Σ (red line) for $c_0 = 0.45$, $C_0 = 3.1$

Proof

$$\begin{aligned}\tilde{S}(x,\tau) - S(x,\tau) &= P(x,\tau) * \tilde{S}_0(x) - p(x,\tau) * N_0(x)\\ &\geq P(x,\tau) * \tilde{S}_0(x) - |p(x,\tau)| * |S_0|\\ &\geq P(x,\tau) * \tilde{S}_0(x) - P(x,\tau) * |S_0|(x) = P(x,\tau)(\tilde{S}_0(x) - |S_0|(x)).\end{aligned} \tag{64}$$

Then and considering $\tilde{S}_0(x) \geq |S_0|(x)$:

$$\tilde{S}(x,\tau) \geq S(x,\tau). \tag{65}$$

for any $0 < \tau < t < \infty$. □

4 Conclusions

The problem P (1) has been discussed showing existence, uniqueness and positivity results. The introduction of a high-order operator has been shown to provide oscillatory solutions as an inherent property. The independent terms of cooperative and Fisher-KPP types keep the oscillatory character of solutions, nonetheless this is not the case when introducing the advection term. Indeed, it has been shown that there exists a time degenerate (not defined for $\tau = 0$) maximum bound for the advection term $c(\tau)$ so that the solution is kept positive. This positivity condition is of relevance in high-order operators theory. If instabilities induce solutions to be negative, the formulation of a comparison principle does not hold in general. Nonetheless, along with the presented analysis, such comparison principle has been shown to exist for a positive asymptotic kernel that has been precisely assessed with the help of a computational exercise.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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